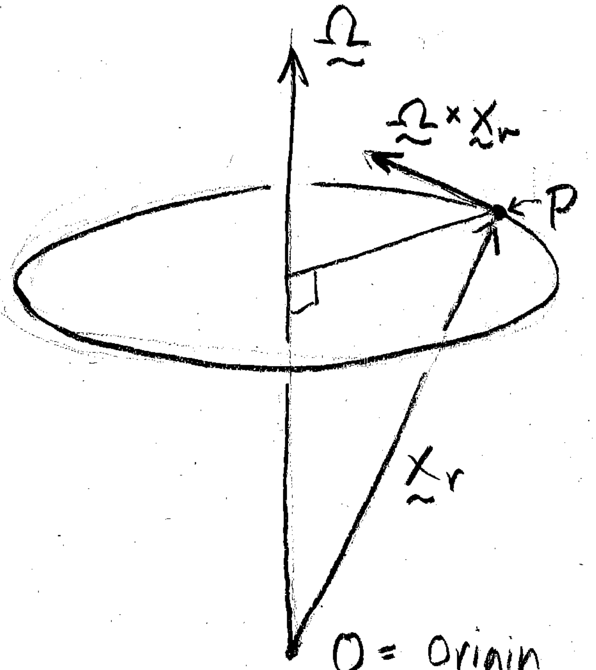
2.3

1/2012

Momentum in a Rotating Frame of Reference (f.o.r.)



- Imagine point P moves in a circle with angular velocity $\boldsymbol{\Omega}$
- Define two frames of reference:

(1) "f" fixed f.o.r. → P moves in a circle, with vector position $\mathbf{x}_f$

(2) "r" f.o.r. rotating with angular velocity $\boldsymbol{\Omega}$. P is motionless, with vector position $\mathbf{x}_r$.

These are related by $\dot{\mathbf{x}}_f = \boldsymbol{\Omega} \times \mathbf{x}_r$ [notation $\frac{d()}{dt} = (\dot{\ })$]

More generally, if the point is <u>moving</u> so that it is not stationary in the moving f.o.r., then

$$\dot{\mathbf{x}}_f = \dot{\mathbf{x}}_r + \boldsymbol{\Omega} \times \mathbf{x}_r \qquad (*)$$

Note: the rotating f.o.r. still rotates with $\boldsymbol{\Omega}$, even if $\mathbf{x}_r$ has a different angular velocity.

Operation (*) applies to <u>any</u> vector, so let's apply it to $\dot{\mathbf{x}}_f$:

$$\Rightarrow \ddot{\mathbf{x}}_f = \frac{d}{dt}\left[\dot{\mathbf{x}}_r + \boldsymbol{\Omega}\times\mathbf{x}_r\right] + \boldsymbol{\Omega}\times\left[\dot{\mathbf{x}}_r + \boldsymbol{\Omega}\times\mathbf{x}_r\right]$$

so, defining $\dot{\mathbf{x}}_f = \mathbf{u}_f$ and $\dot{\mathbf{x}}_r = \mathbf{u}_r$, this is

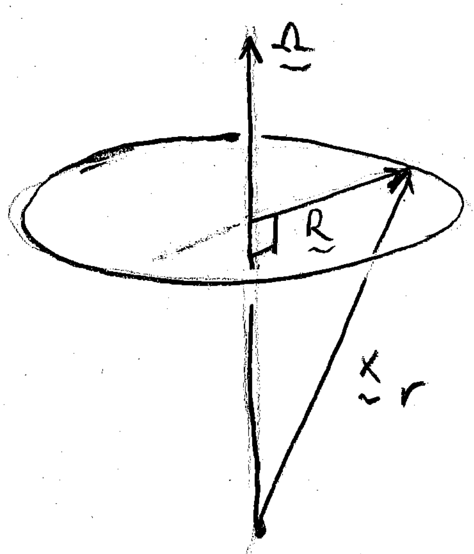
$$\boxed{\dot{\mathbf{u}}_f = \dot{\mathbf{u}}_r + 2\boldsymbol{\Omega}\times\mathbf{u}_r + \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{x}_r}$$

<u>Coriolis</u> acceleration + <u>centripetal</u> acceleration

<u>apparent</u> acceleration in rotating f.o.r.

<u>actual</u> acceleration in fixed f.o.r.

• Note that if we define normal vector $\mathbf{R}$



then $\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{R} = -\Omega^2\mathbf{R}$

where $|\boldsymbol{\Omega}| \equiv \Omega$

(points toward the center of the circle)

Now we can rewrite $\boxed{\mathbf{x} \text{ mom}}$ in the rotating f.o.r., treating point P as a fluid parcel.

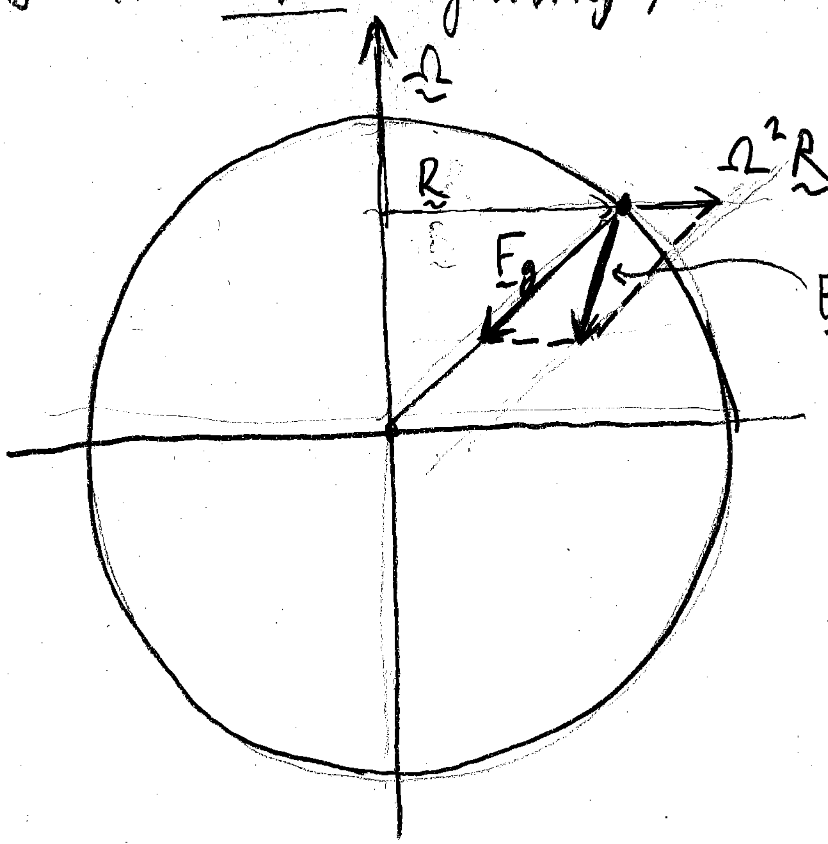
force of gravity / unit mass

$$\boxed{\mathbf{x} \text{ mom}}_f \qquad \frac{D\mathbf{u}_f}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F}_g$$

$$\boxed{\mathbf{x} \text{ mom}}_r \qquad \frac{D\mathbf{u}_r}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u}_r = -\frac{1}{\rho}\nabla p + \mathbf{F}_g + \Omega^2\mathbf{R}$$

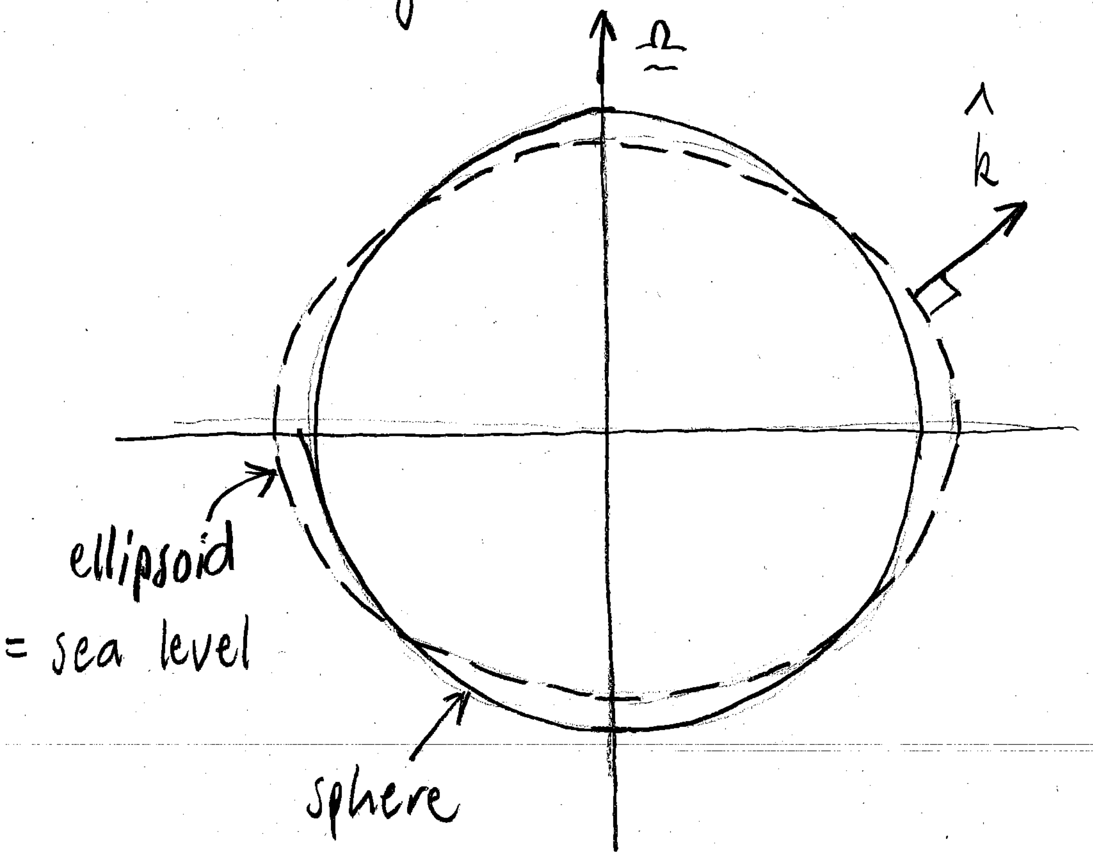
"Apparent" $\frac{\text{Forces}}{\text{unit mass}}$: Coriolis and Centrifugal

On a rotating planet we combine the last two terms to <u>define</u> gravity, and the direction "up" ($\hat{k}$)



$$\mathbf{F}_g + \Omega^2\mathbf{R} \equiv -g\hat{k}$$

Earth deforms into an ellipsoid due to the centrifugal force



Radius differs from spherical by ~ 42 km out of 6371 km = 0.7 %

Recall the geopotential $\Phi = gz$ ($z = 0$ at sea level)

so $\quad \mathbf{F}_g + \Omega^2 \mathbf{R} = -g\hat{k} = -\nabla \Phi$

Thus, dropping the "r" subscript, $\boxed{\mathbf{x} \text{ mom}}$ for an observer on the rotating Earth is

$$\boxed{\mathbf{x} \text{ mom}} \quad \boxed{\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p - g\hat{k}}$$

new term