The horizontal pressure gradient

\[ \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{2\pi}{\epsilon} \frac{\partial f}{\partial x} \]

Highly simplified momentum equations:
- linear (no \( u \cdot \nabla u \))
- non rotating
- no friction
- hydrostatic

Schematically:

- Force per unit mass due to pressure

Procedure to get to \( \frac{\partial p}{\partial x} \):

1. Use \( x_{\text{max}} \) and knowledge of \( p(x) \) to get \( p(x) \)
2. Then calculate \( \frac{\partial p}{\partial x} \) to get \( \frac{\partial u}{\partial x} \)
**Example OCEAN:**

\[ \eta(x) = \text{free surface} \]

\[ z = 0 \]

\[ \rho = \text{const. contours} \]

\[ \text{What is } \rho(x, z)? \]

\[ \frac{\partial \rho}{\partial z} = -\rho g \]

\[ \Rightarrow \int_{z}^{\eta} \frac{\partial \rho}{\partial z} \, d\hat{z} = -g \int_{z}^{\eta} \rho \, d\hat{z} \]

Assume \( \rho = \rho_0 + \rho' \), with \( \rho_0 = \text{const.} = 1000 \text{ kg m}^{-3} \)

\[ \Rightarrow [\rho'] \ll [\rho_0] \quad \Rightarrow [\text{scale of }] \]

\[ \Rightarrow \rho(\eta) - \rho(z) = \rho_{\text{atm}} - \rho \]

\[ = -g \rho_0 (\eta - z) - g \int_{z}^{\eta} \rho' \, d\hat{z} \]
or
\[ p = \rho_{\text{atm}} + g \rho_0 (\eta - z) + g \int_z^n \frac{\nabla p'}{\nabla z} \, d\hat{z} \]
\begin{align*}
\text{weight of atm.} & \quad \text{weight of ocean} \\
\text{unit area} & \quad \text{unit area}
\end{align*}

Next take \( \frac{\partial}{\partial x} \), and assume \( \rho_{\text{atm}} = \text{const.} \), so \( \frac{\partial}{\partial x} \rho_{\text{atm}} = 0 \)

\[ \Rightarrow \frac{\partial}{\partial x} \int_{z}^{n} \nabla p' \, d\hat{z} = \frac{\partial}{\partial x} \left[ \frac{\rho_0}{\nabla z} \right] \left[ \frac{\nabla p'}{\nabla z} \right] \quad \text{due to tilt of isopycnals} \]
\[ \quad \text{due to tilt of free surface} \]

and note that we can use a power series expansion to simplify \( \frac{1}{\epsilon} = \frac{1}{\rho_0 + \epsilon'} = \frac{1}{\rho_0} \left[ 1 - \frac{\epsilon'}{\rho_0} + \left( \frac{\epsilon''}{\rho_0} \right)^2 + \ldots \right] \)

and for \( \frac{\epsilon'}{\rho_0} \ll 1 \) we keep only the leading term so \( \frac{1}{\epsilon} \approx \frac{1}{\rho_0} \)

so we can write:

\[ \boxed{\text{man} \frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x} - g \frac{\partial}{\partial x} \int_{z}^{n} \frac{p'}{\rho_0} \, d\hat{z}} \]

used this in our lab experiment
Example ATMOSPHERE:

- we can no longer assume \( p = \text{const.} \).

Instead, we keep track of the heights of constant pressure surfaces (the geopotential).

Consider two ways to calculate \( \delta p \).

First way:

\[
\delta p = \left. \frac{\partial p}{\partial x} \right|_z \delta x.
\]

\[
\Rightarrow \quad \frac{\delta p}{\delta x} = \left. \frac{\partial p}{\partial x} \right|_z \quad (\text{first term in Taylor series expansion})
\]
Second way:

\[
\delta P = \left. \frac{\partial P}{\partial z} \right|_{x} (-\delta z)
\]

Note minus sign!

\[
\Rightarrow \frac{\delta P}{\delta x} = \rho g \frac{\delta z}{\delta x}
\]

By \( z \text{man} \) hydrostatic

then, in order to calculate \( \delta z \) we our (assumed) knowledge of the shape of pressure surfaces:

\[
\delta z = \left. \frac{\partial z}{\partial x} \right|_{p} \delta x \Rightarrow \frac{\delta z}{\delta x} = \left. \frac{\partial z}{\partial x} \right|_{p}
\]

\[\therefore \frac{\delta P}{\delta x} = \rho g \left( \frac{\partial z}{\partial x} \right) \mid_{p} \text{ (II)}\]

Combining (I) and (II)

\[
\Rightarrow -\left. \frac{1}{\rho} \frac{\partial P}{\partial x} \right|_{z} = -\rho \left. \frac{\partial z}{\partial x} \right|_{p}
\]
and we define the "geopotential" $\Phi = \phi$

$$\Phi = \int_0^z g \, dz = g \, z$$  \hspace{1cm} \text{(assume } g = \text{const.)}$$

$\text{sea level}$

$$-\frac{1}{\rho} \left. \frac{\partial \Phi}{\partial x} \right|_z = -\left. \frac{\partial \Phi}{\partial x} \right|_p$$

where $\Phi |_p$ is $g$ times the height of a given pressure surface, e.g. 500 mb

$$\Rightarrow \quad \boxed{\text{mm}} \quad \frac{\partial u}{\partial t} = -\left. \frac{\partial \Phi}{\partial x} \right|_p$$

* removes explicit dependence on $\rho$

* looks a lot like $\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}$