2.1
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# The horizontal pressure gradient

$\boxed{x \text{ mom}}$ $\quad \dfrac{\partial u}{\partial t} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$

$\boxed{z \text{ mom}}$ $\quad 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g$

Highly simplified momentum equations:
- linear (no $\underset{\sim}{u} \cdot \nabla u$)
- non rotating
- no friction
- hydrostatic

Schematically:


$\dfrac{\text{force}}{\text{unit mass}}$ due to pressure

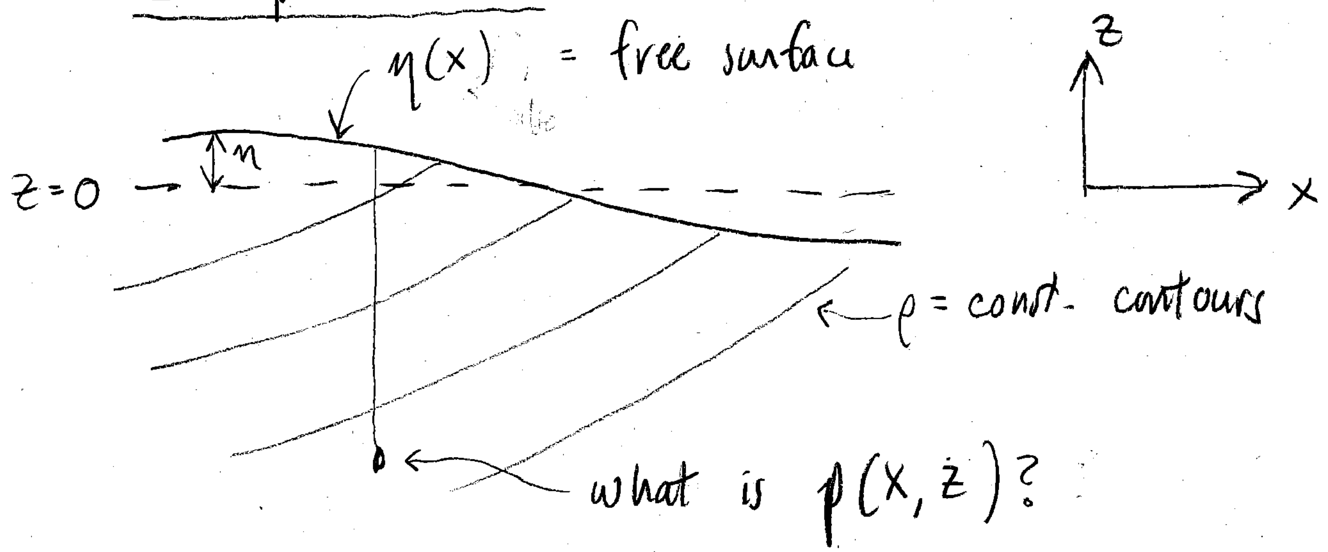
Notation $\underset{\sim}{x} = (x, y, z)$

Procedure to get to $\partial p / \partial x$:

(i) use $\boxed{z \text{ mom}}$ and knowledge of $\rho(\underset{\sim}{x})$ to get $p(\underset{\sim}{x})$

(ii) then calculate $\dfrac{\partial p}{\partial x}$ to get $\dfrac{\partial u}{\partial t}$

Example OCEAN:




$\boxed{z \text{ mom}}$ $\frac{\partial p}{\partial z} = -\rho g$

$$\Rightarrow \int_z^\eta \frac{\partial p}{\partial \hat{z}}\, d\hat{z} = -g \int_z^\eta \rho \, d\hat{z}$$

assume $\rho = \rho_0 + \rho'$, with $\rho_0$ = const. = 1000 kg $m^{-3}$

$$\Rightarrow [\rho'] \ll [\rho_0]$$

[ ] ≡ "scale of"

$$\Rightarrow p(\eta) - p(z) = p_{atm} - p$$

$$= -g\rho_0(\eta - z) - g\int_z^\eta \rho' \, d\hat{z}$$

or

$$p = \underbrace{p_{atm}}_{\frac{\text{weight of atm.}}{\text{unit area}}} + \underbrace{g\rho_0(\eta - z) + g\int_z^\eta \rho' \, d\hat{z}}_{\frac{\text{weight of ocean}}{\text{unit area}}}$$

Next take $\frac{\partial}{\partial x}$, and assume $p_{atm} = \text{const.}$, so $\frac{\partial p_{atm}}{\partial x} = 0$

$$\Rightarrow \frac{\partial p}{\partial x} = g\rho_0 \frac{\partial \eta}{\partial x} + g\frac{\partial}{\partial x}\int_z^\eta \rho' \, d\hat{z}$$

← due to tilt of isopycnals

↑ due to tilt of free surface

• and note that we can use a power series expansion to simplify $\frac{1}{\rho} = \frac{1}{\rho_0 + \rho'} = \frac{1}{\rho_0}\frac{1}{(1 + \rho'/\rho_0)} = \frac{1}{\rho_0}\left[1 - \frac{\rho'}{\rho_0} + \left(\frac{\rho'}{\rho_0}\right)^2 - \dots\right]$

and for $\frac{\rho'}{\rho_0} \ll 1$ we keep only the leading term so $\frac{1}{\rho} \cong \frac{1}{\rho_0}$

so we can write:

x mom

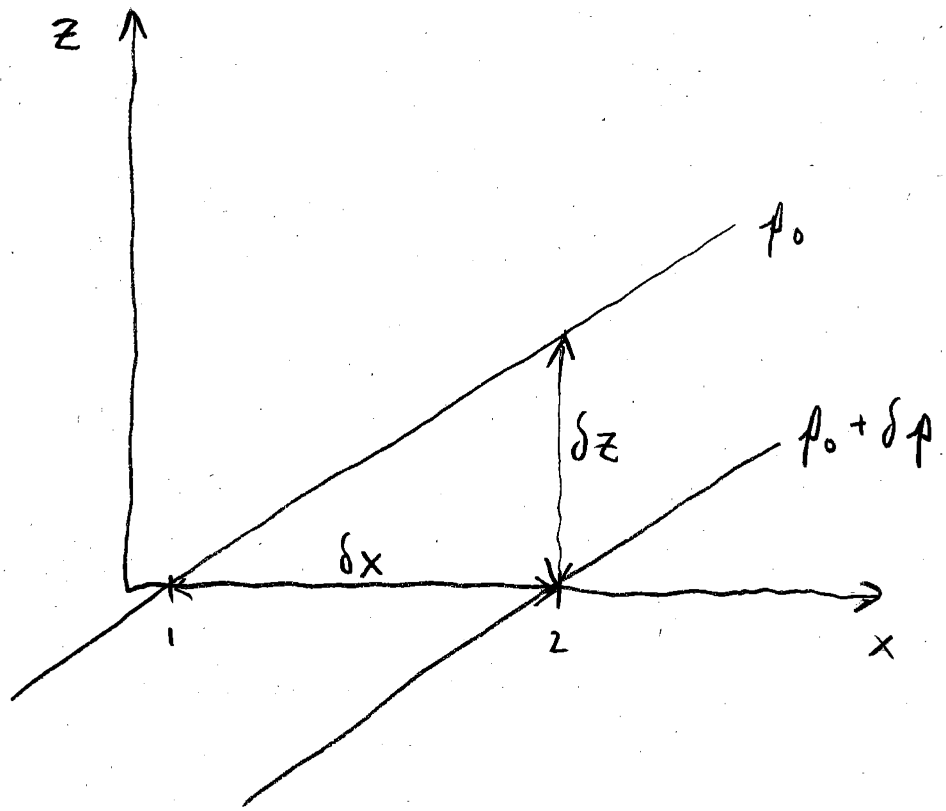
$$\boxed{\underbrace{\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}}_{\text{used this in our lab experiment}} - g\frac{\partial}{\partial x}\int_z^\eta \frac{\rho'}{\rho_0} \, d\hat{z}}$$

Example ATMOSPHERE:

- we can no longer assume $\rho \sim$ const.

Instead: we keep track of the heights of constant pressure surfaces (the geopotential)

Consider two ways to calculate $\delta p$




First way:

$$\delta p = \left.\frac{\partial p}{\partial x}\right|_z \delta x$$

(first term in Taylor series expansion)

$$\Rightarrow \quad \frac{\delta p}{\delta x} = \left.\frac{\partial p}{\partial x}\right|_z \qquad (\mathrm{I})$$

Second way:

$$\delta p = \underbrace{\left.\frac{\partial p}{\partial z}\right|_x}_{= -\rho g \text{ by } \boxed{z \text{ mom}} \text{ hydrostatic}} (-\delta z) \qquad \text{Note minus sign!}$$

$$\Rightarrow \quad \frac{\delta p}{\delta x} = \rho g \frac{\delta z}{\delta x}$$

then, in order to calculate $\delta z$ we our (assumed) knowledge of the shape of pressure surfaces:

$$\delta z = \left.\frac{\partial z}{\partial x}\right|_p \partial x \Rightarrow \frac{\delta z}{\delta x} = \left.\frac{\partial z}{\partial x}\right|_p$$

$$\therefore \quad \frac{\delta p}{\delta x} = \rho g \left.\left(\frac{\partial z}{\partial x}\right)\right|_p \qquad \text{II}$$

Combining I and II

$$\Rightarrow \quad -\frac{1}{\rho}\left.\frac{\partial p}{\partial x}\right|_z = -g\left.\frac{\partial z}{\partial x}\right|_p$$

and we define the "geopotential" $\Phi$ "Phi"

$$\Phi \equiv \int_0^z g\, d\hat{z} = gz \qquad (\text{assume } g = \text{const.})$$

$0 \leftarrow$ sea level

$$\therefore \quad -\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_z = -\frac{\partial \Phi}{\partial x}\bigg|_p$$

where $\Phi|_p$ is $g$ times the height of a given pressure surface, e.g. 500 mb

$$\Rightarrow \quad \boxed{\text{x mom}} \quad \boxed{\frac{\partial u}{\partial t} = -\frac{\partial \Phi}{\partial x}\bigg|_p}$$

★ removes explicit dependence on $\rho$

★ looks a lot like $\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$