### Properties of Air & Water

#### Equation of State

- **Potential Temperature & Density**
- **Buoyancy, Frequency & Compressibility**

<table>
<thead>
<tr>
<th>Air</th>
<th>Seawater</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_2: 78.1%$, $O_2: 21.0%$, ...</td>
<td>$H_2O$, $NaCl$, ...</td>
</tr>
<tr>
<td>$p = p(T, \rho, q)$ [\text{Eqn. of state}] $p = p(s, T, \rho)$</td>
<td>$\text{Specific humidity} \propto \frac{\text{mass of water vapor}}{\text{mass of moist air}}$</td>
</tr>
<tr>
<td>$p \to 0 \text{ kg m}^{-3}$ [\text{top of atm.}]</td>
<td>$\rho \sim 1000 \text{ kg m}^{-3}$ [\text{freshwater}]</td>
</tr>
<tr>
<td>$p \sim \frac{1.2}{e} \text{ kg m}^{-3}$ [\text{at 7-8 km (scale height)}]</td>
<td>$\rho \sim 1025 \text{ kg m}^{-3}$ [\text{ocean surface}]</td>
</tr>
<tr>
<td>$p \sim 1.2 \text{ kg m}^{-3}$ [\text{sea level}]</td>
<td>$\rho \sim 1055 \text{ kg m}^{-3}$ [\text{abyss, largely due to compressibility}]</td>
</tr>
</tbody>
</table>

- Varies by $\mathcal{O}(1)$
- Varies by only $\sim 5\%$
More about seawater $p = p(s, T, p)$.

- Most important density variation is due to temperature, although salinity is important in coastal or polar regions.

**Typical Values** - at $T = 10^\circ C$, $S = 35$; units are 'practical salinity' = psu

- $p = 0$
- $\Delta p = 1027$ kg m$^{-3}$

with rates of change:

$$\alpha = -\frac{1}{p} \left. \frac{\partial p}{\partial T} \right|_p = 1.168 \times 10^{-7} \text{ K}^{-1}, \quad \frac{\partial p}{\partial S} = 0.781 \text{ kg m}^{-3} \text{ (per psu)}$$

(These vary with $T$, $S$, $p$.)

So: increasing salinity by 1 psu $\Rightarrow \Delta p \approx 0.78$ kg m$^{-3}$

Increasing temperature by 1$^\circ$C $\Rightarrow \Delta p \approx -0.12$ kg m$^{-3}$

Speed of sound: $c_s = \sqrt{\frac{\partial p}{\partial T}} = 1500$ m s$^{-1}$

Note: Often the dynamically important density changes are just a few kg m$^{-3}$!
More about air - neglecting moisture  \( p = p(T, p, x) \)

Dry air - an "ideal gas" \( \Rightarrow p = \rho RT \) (\(*\))

\( R = \text{gas constant} = 287 \text{ J kg}^{-1} \text{ K}^{-1} \)

\( ^0\text{K} = 273.15 = ^\circ\text{C} \)

What if we compress air in a cylinder?

\[
\begin{array}{c}
\begin{array}{c}
V_1 \\
T_1 \\
p_1
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
V_2 \\
T_2 \\
p_2
\end{array}
\end{array}
\]

\( V_2 = V_1/2 \)

Clearly \( p_2 = 2 \times p_1 \), but what about \( p \circ T \)?

Expect both to increase, but how much?

\( \Rightarrow \text{Need another equation to use with (\(*\))} \)

From thermodynamics:

If the change is (i) "adiabatic" (no heat added) and (ii) "reversible" (e.g. no viscous dissipation) then the change is "isentropic" and

\[
\frac{p}{\rho^\gamma} = \text{const.} \quad (\ast \ast) \]

where \( \gamma = 1.4 \) for dry air

"gamma"
so for isentropic changes of an ideal gas from \( p \) to \( p_r \) \((r = \text{"reference"})\)\
\[
(**) \Rightarrow \frac{p}{p^*} = \frac{p_r}{p_r^*}
\]
then, using \( p = pRT \) it is easy to show:

\[
\left\{ \begin{array}{c}
\frac{T}{T_r} = \left( \frac{p_r}{p} \right)^{\frac{\gamma - 1}{\gamma}} \\
\frac{p_r}{\rho} = \left( \frac{p_r}{p} \right)^{\frac{1}{\gamma}}
\end{array} \right.
\]

use *K*!

and the usual notation is

\[
\begin{align*}
T_r &= \Theta = \text{potential temperature} \\
p_r &= \rho_{\text{pot}} = \text{potential density}
\end{align*}
\]

These quantities are used for seawater too!

There are the temperature & density after an isentropic change of pressure to \( p_r \).
Next, re-consider the **Buoyancy Frequency**, but for a compressible fluid.

\[ \Delta e \] is most easily approximated by considering isentropic displacement of ambient fluid to \( \rho_r \)

and thus \( \Delta e = -\Delta z \frac{d \rho_{pot}}{d z} \)

(note \( \rho_{pot} = \rho_{amb} \) at \( z_r \), where \( \rho = \rho_r \))

Hence \( N = \sqrt{-\frac{g}{\rho_r} \frac{d \rho_{pot}}{d z}} \) a better estimate of the Buoyancy Frequency

(most accurate near \( z_r \))
Typical value of \( N \)

Lower stratosphere \( 1.7 \times 10^{-2} \ s^{-1} \)

Troposphere \( 10^{-1} \ s^{-1} \) \( \frac{2\pi}{N} = 10 \text{ minutes} \)

Upper Ocean \( 10^{-2} \ s^{-1} \)

Deep Ocean \( 10^{-3} \ s^{-1} \)

Why does \( N \) increase upwards in both cases? 

{prompted interesting discussion! save time for it}

Q: Why use potential density?

A: A fluid with \( \rho_{\text{pot}} = \text{const.} \) is not stratified 

even though it has \( \frac{\partial \rho_{\text{pot}}}{\partial z} < 0 \)