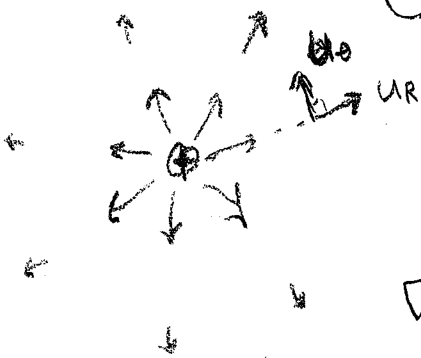
Fundamental 2D Potential Flow Solutions — must satisfy $\nabla^2 \varphi = 0$

① Point source: $u_R = \frac{m}{2\pi}\frac{1}{r}$



$\left[\text{"strength"} = m = \frac{\text{volume flux}}{\text{unit z thickness}}\right] \frac{m^3}{s\, m} \sim \frac{m^2}{s}$

$$\nabla\varphi = \hat{i}_R \frac{\partial\varphi}{\partial r} + \hat{i}_\theta \frac{1}{r}\cancel{\frac{\partial\varphi}{\partial\theta}} = \hat{i}_R u_R + \cancel{\hat{i}_\theta u_\theta}$$

$$\Rightarrow \frac{\partial\varphi}{\partial r} = \frac{m}{2\pi}\frac{1}{r} \Rightarrow \boxed{\varphi = \frac{m}{2\pi}\ln r}$$

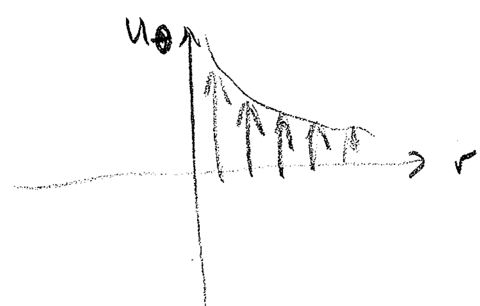
dimensionless, but if $m \sim \frac{m^2}{s}$ then $r \sim m$ ...

(point sink if m is negative)

② Uniform flow $\underset{\sim}{u} = (U, 0, 0)$

$$\Rightarrow \frac{\partial\varphi}{\partial x} = U \quad \text{so} \quad \boxed{\varphi = Ux}$$

③ Point Vortex



$u_\theta = \frac{\Gamma}{2\pi}\frac{1}{r}$

$u_R = 0$

easy to show $\varphi = \frac{\Gamma}{2\pi}\theta$

$\left(u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$

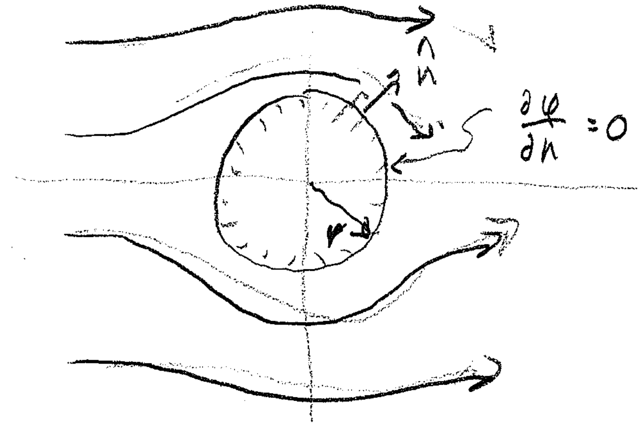
2D Flow around a cylinder

- combine: source + sink + uniform flow $= \varphi_1 + \varphi_2 + \varphi_3 = \varphi$

to get solution for flow around a cylinder!

$U \Rightarrow$

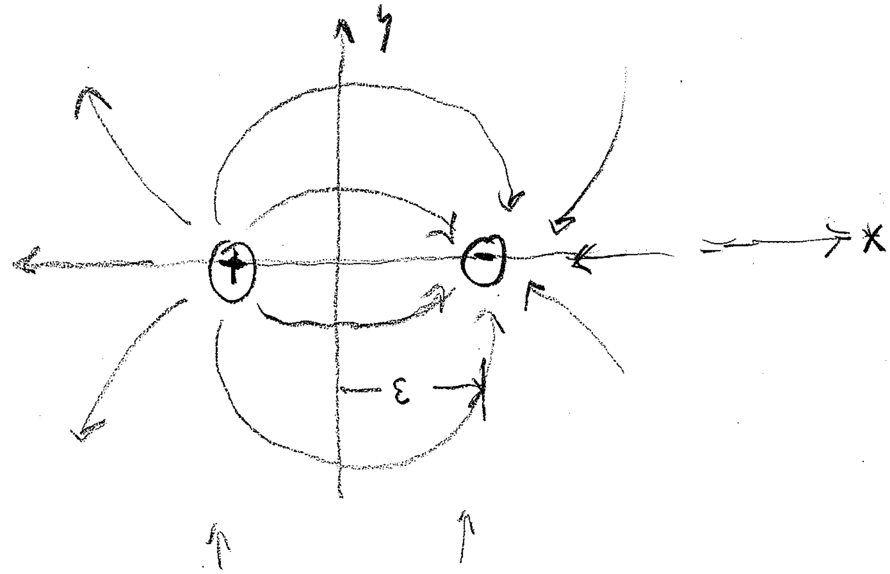
$\underline{u} \to (U,0,0)$ as $r \to \infty$



$\frac{\partial \varphi}{\partial n} = 0$ on $r = a$

Recipe:

(1) Combine a source + sink in a "doublet"



$$\varphi_1 + \varphi_2 = \frac{m}{2\pi} \ln r^+ - \frac{m}{2\pi} \ln r^-$$

(source) (sink)

where $r^+ \equiv \sqrt{(x+\varepsilon)^2 + y^2}$, $r^- \equiv \sqrt{(x-\varepsilon)^2 + y^2}$

then take limit $\varepsilon \to 0$ but m increases

so $\frac{m\varepsilon}{\pi} \to \mu$ (a constant)

- use power series expansions and drop terms $\mathcal{O}(\varepsilon^2)$ (compared to ε)

$$\varphi_1 + \varphi_2 = \frac{m}{2\pi}\left(\ln r^+ - \ln r^-\right) = \frac{m}{2\pi} \ln\left(\frac{r^+}{r^-}\right)$$

Note $r^+ = \sqrt{x^2 + 2\varepsilon x + \cancel{\varepsilon^2} + y^2} \cong \sqrt{r^2 + 2\varepsilon x} = r\sqrt{1 + \frac{2\varepsilon x}{r^2}}$

$$= r\left[1 + \frac{1}{2}\frac{2\varepsilon x}{r^2} + \cancel{\mathcal{O}(\varepsilon^2)}\right] \cong r + \frac{\varepsilon x}{r}$$

so $\frac{r^+}{r^-} \cong \frac{r + \varepsilon x/r}{r - \varepsilon x/r} = \frac{1 + \varepsilon x/r^2}{1 - \varepsilon x/r^2} = \left(1 + \frac{\varepsilon x}{r^2}\right)\left[1 + \frac{\varepsilon x}{r^2} + \cancel{\mathcal{O}(\varepsilon^2)}\right]$

$$\cong 1 + 2\varepsilon x/r^2 + \cancel{\mathcal{O}(\varepsilon^2)}$$

so

$$\ln\left(\frac{r^+}{r^-}\right) \cong \ln\left(1 + \frac{2\varepsilon x}{r^2}\right) = \frac{2\varepsilon x}{r^2} - \cancel{\mathcal{O}(\varepsilon^2)}$$

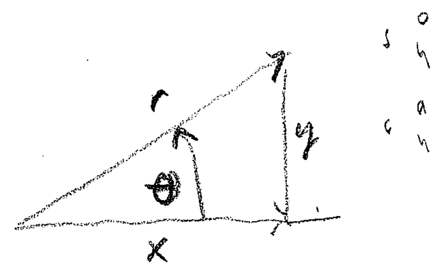
so

$$\varphi_1 + \varphi_2 = \lim_{\substack{\varepsilon \to 0 \\ m\varepsilon \to \mu}} \left[\frac{m}{2\pi} \ln\left(\frac{r^+}{r^-}\right)\right] = \frac{m}{\cancel{2}\pi}\frac{\cancel{2}\varepsilon x}{r^2} = \frac{\mu x}{r^2}$$

(2) add a uniform flow $\varphi_3 = \mathcal{U}x$

(3) Ensure no flow through cylinder wall $\Rightarrow \frac{\partial \varphi}{\partial r} = 0$ on $r = a$

rewrite in polar coordinates:



$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 = \mathcal{U}x + \frac{\mu x}{r^2}$$

$x = r\cos\theta$

$$\varphi = r\cos\theta\left(\mathcal{U} + \frac{\mu}{r^2}\right)$$

$$\frac{\partial \varphi}{\partial r} = \cos\theta\left(\mathcal{U} + \frac{\mu}{r^2}\right) + r\cos\theta\left(-\frac{2\mu}{r^3}\right)$$

$$\text{B.C.} \Rightarrow \mathcal{U} + \frac{\mu}{a^2} - \frac{2\mu}{a^2} = 0 \Rightarrow \mathcal{U} = \frac{\mu}{a^2}$$

so $\mu = a^2\mathcal{U}$

$$\varphi = r\cos\theta\left(\mathcal{U} + \frac{a^2\mathcal{U}}{r^2}\right) = \boxed{\mathcal{U}\left(r + \frac{a^2}{r}\right)\cos\theta = \varphi}$$

Table of power series expansions

(In some cases the remainder term $R_n(x)$ is given)

(B_n=Bernoulli numbers, E_n=Euler numbers, see sec. 12.3)

Function	Power series expansion	Interval of convergence
	Algebraic functions $\binom{\alpha}{n}=\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$, α real number	
$(1+x)^\alpha$	$1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\ldots+\binom{\alpha}{n}x^n+\ldots,$	$-1<x<1$
	$R_n(x)=\binom{\alpha}{n}(1+\theta x)^{\alpha-n}x^n,\ 0<\theta<1$	
$\frac{1}{1-x}$	$1+x+x^2+x^3+\ldots+x^n+\ldots$	$-1<x<1$
$\frac{1}{1+x}$	$1-x+x^2-x^3+\ldots+(-1)^n x^n+\ldots$	$-1<x<1$
$\frac{1}{a-bx}$	$\frac{1}{a}\left[1+\frac{bx}{a}+\left(\frac{bx}{a}\right)^2+\ldots+\left(\frac{bx}{a}\right)^n+\ldots\right]$ or	$\lvert x\rvert<\left\lvert\frac{a}{b}\right\rvert$
	$-\frac{1}{bx}\left[1+\frac{a}{bx}+\left(\frac{a}{bx}\right)^2+\ldots+\left(\frac{a}{bx}\right)^n+\ldots\right]$	$\lvert x\rvert>\left\lvert\frac{a}{b}\right\rvert$
$\frac{1}{(1-x)^2}$	$1+2x+3x^2+\ldots+(n+1)x^n+\ldots$	$-1<x<1$
$\sqrt{1+x}$	$1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\ldots+\binom{1/2}{n}x^n+\ldots$	$-1\leqslant x\leqslant 1$
$\frac{1}{\sqrt{1+x}}$	$1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}+\frac{35x^4}{128}-\ldots+\binom{-1/2}{n}x^n+\ldots$	$-1<x\leqslant 1$

Note: $\left\lvert\binom{\alpha}{n}\right\rvert\sim C_\alpha n^{-\alpha-1},\ n\to\infty$

Table of fractional binomial coefficients, see below.

	Exponential, hyperbolic, logarithmic and inverse hyperbolic functions	
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$	$-\infty<x<\infty$
	$R_n(x)=\frac{e^{\theta x}}{n!}x^n,\ 0<\theta<1$	
a^x	$1+x\ \ln a+\frac{(x\ \ln a)^2}{2!}+\ldots+\frac{(x\ \ln a)^n}{n!}+\ldots$	$-\infty<x<\infty$

$\frac{1}{e^x-1}$	$\frac{1}{x}-\frac{1}{2}+\frac{x}{12}-\frac{x^3}{30\cdot 4!}+\ldots+\frac{B_{2n}\,x^{2n-1}}{(2n)!}+\ldots$	$-2\pi<x<2\pi,\ x\neq 0$
$\sinh x$	$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots+\frac{x^{2n+1}}{(2n+1)!}+\ldots$	$-\infty<x<\infty$
$\cosh x$	$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots$	$-\infty<x<\infty$
$\tanh x$	$x-\frac{x^3}{3}+\frac{2x^5}{15}-\frac{17x^7}{315}+\ldots+\frac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}x^{2n-1}+\ldots$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$
$\coth x$	$\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\frac{2x^5}{945}+\ldots+\frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1}+\ldots$	$-\pi<x<\pi,\ x\neq 0$
$\frac{1}{\sinh x}$	$\frac{1}{x}-\frac{x}{6}+\frac{7x^3}{360}-\ldots-\frac{2^{2n}-2}{(2n)!}B_{2n}x^{2n-1}+\ldots$	$-\pi<x<\pi,\ x\neq 0$
$\frac{1}{\cosh x}$	$1-\frac{x^2}{2}+\frac{5x^4}{24}-\ldots+\frac{E_{2n}}{(2n)!}x^{2n}+\ldots$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$	$-1<x\leq 1$
	$R_n(x)=\frac{(-1)^{n-1}}{1+\theta x}\cdot\frac{x^n}{n},\ 0<\theta<1$	
$\ln(a+x)$	$\ln a+\frac{x}{a}-\frac{1}{2}\left(\frac{x}{a}\right)^2+\frac{1}{3}\left(\frac{x}{a}\right)^3-\ldots+\frac{(-1)^{n-1}}{n}\left(\frac{x}{a}\right)^n+\ldots$	$-a<x\leq a$
$\ln(1+x)$	$\frac{x}{1+x}+\frac{1}{2}\left(\frac{x}{1+x}\right)^2+\ldots+\frac{1}{n}\left(\frac{x}{1+x}\right)^n+\ldots$	$x>-\frac{1}{2}$
$\operatorname{arsinh} x$	$x-\frac{x^3}{6}+\frac{3x^5}{40}-\ldots+(-1)^n\cdot\frac{(2n-1)!!}{(2n)!!}\,\frac{x^{2n+1}}{2n+1}+\ldots$	$-1<x<1$
$\operatorname{arcosh} x$	$\ln\lvert 2x\rvert-\frac{1}{4x^2}-\frac{3}{32x^4}-\ldots-\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2nx^{2n}}-\ldots$	$\lvert x\rvert>1$
$\operatorname{artanh} x$	$x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots+\frac{x^{2n+1}}{2n+1}+\ldots$	$-1<x<1$
$\operatorname{arcoth} x$	$\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\ldots+\frac{1}{(2n+1)x^{2n+1}}+\ldots$	$\lvert x\rvert>1$

	Trigonometric and inverse trigonometric functions	
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots$	$-\infty<x<\infty$
	$R_{2n+1}(x)=(-1)^n\frac{\cos\theta x}{(2n+1)!}x^{2n+1},\ 0<\theta<1$	
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots$	$-\infty<x<\infty$
	$R_{2n}(x)=(-1)^n\frac{\cos\theta x}{(2n)!}x^{2n},\ 0<\theta<1$	

$\tan x$

$\cot x$

$\sec x=\frac{1}{\cos x}$

$\csc x=\frac{1}{\sin x}$

$\arcsin x$

$\arctan x$

$\arccos x$

$\operatorname{arccot} x$

Graphs of s[...]

$y=\sin x$

$y=\ln(1+$[...]

$\tan x$	$x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\dfrac{17x^7}{315}+\ldots+(-1)^{n-1}\dfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_{2n}x^{2n-1}+\ldots$	$-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$
$\cot x$	$\dfrac{1}{x}-\dfrac{x}{3}-\dfrac{x^3}{45}-\dfrac{2x^5}{945}-\ldots+(-1)^n\dfrac{2^{2n}}{(2n)!}B_{2n}x^{2n-1}+\ldots$	$-\pi<x<\pi$, $x\neq 0$
$\sec x=\dfrac{1}{\cos x}$	$1+\dfrac{x^2}{2!}+\dfrac{5x^4}{4!}+\dfrac{61x^6}{6!}+\ldots+(-1)^n\dfrac{E_{2n}}{(2n)!}x^{2n}+\ldots$	$-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$
$\csc x=\dfrac{1}{\sin x}$	$\dfrac{1}{x}+\dfrac{x}{3!}+\dfrac{7x^3}{3\cdot 5!}+\dfrac{31x^5}{3\cdot 7!}+\ldots+(-1)^{n-1}\cdot\dfrac{2^{2n}-2}{(2n)!}B_{2n}x^{2n-1}+\ldots$	$-\pi<x<\pi$, $x\neq 0$
$\arcsin x$	$x+\dfrac{x^3}{6}+\dfrac{3x^5}{40}+\ldots+\dfrac{(2n-1)!!}{(2n)!!}\dfrac{x^{2n+1}}{2n+1}+\ldots$	$-1<x<1$
$\arctan x$	$x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots+(-1)^n\dfrac{x^{2n+1}}{2n+1}+\ldots$	$-1\leq x\leq 1$
	$R_{2n+1}(x)=(-1)^n\dfrac{1}{1+\theta^2x^2}\cdot\dfrac{x^{2n+1}}{2n+1}$, $0<\theta<1$	
$\arccos x$	$=\dfrac{\pi}{2}-\arcsin x$	
$\operatorname{arccot} x$	$=\dfrac{\pi}{2}-\arctan x$	

Graphs of some Taylor polynomials $P_n(x)$ of degree n