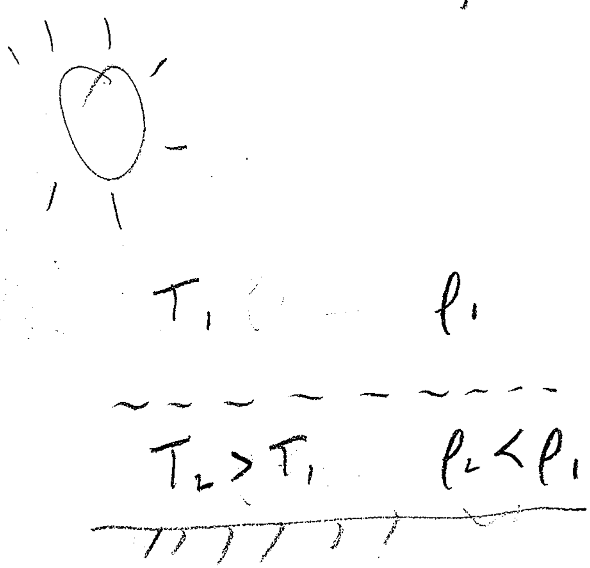
K. Instability

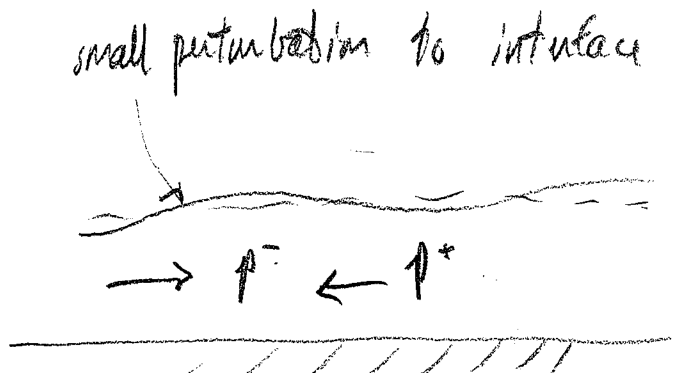
$\boxed{\text{V}.3\text{-}5}$ should be VI.3-5

12/4/2009

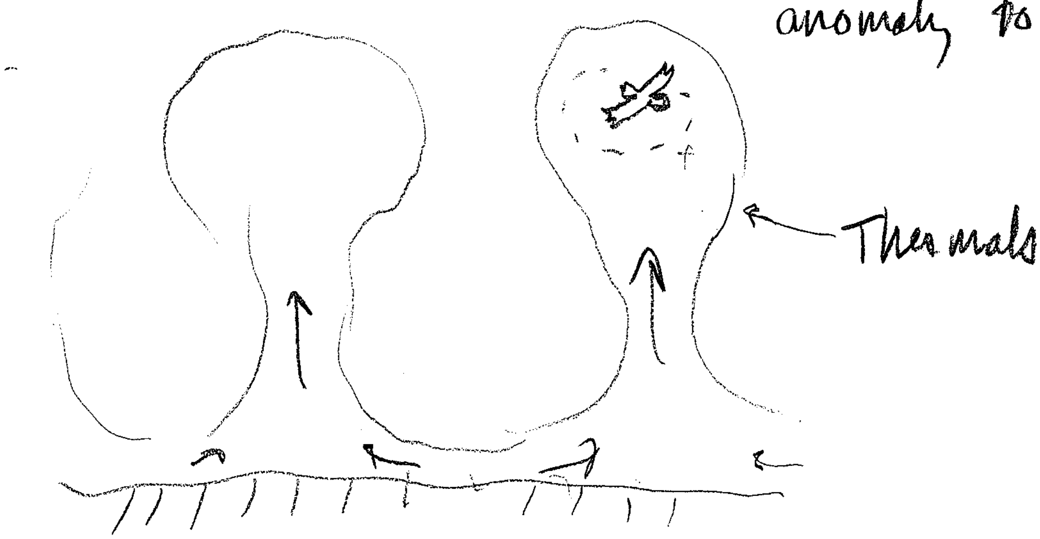
Often a physical fluid state is <u>unstable</u> to small perturbations

e.g. solar heating of air near the ground

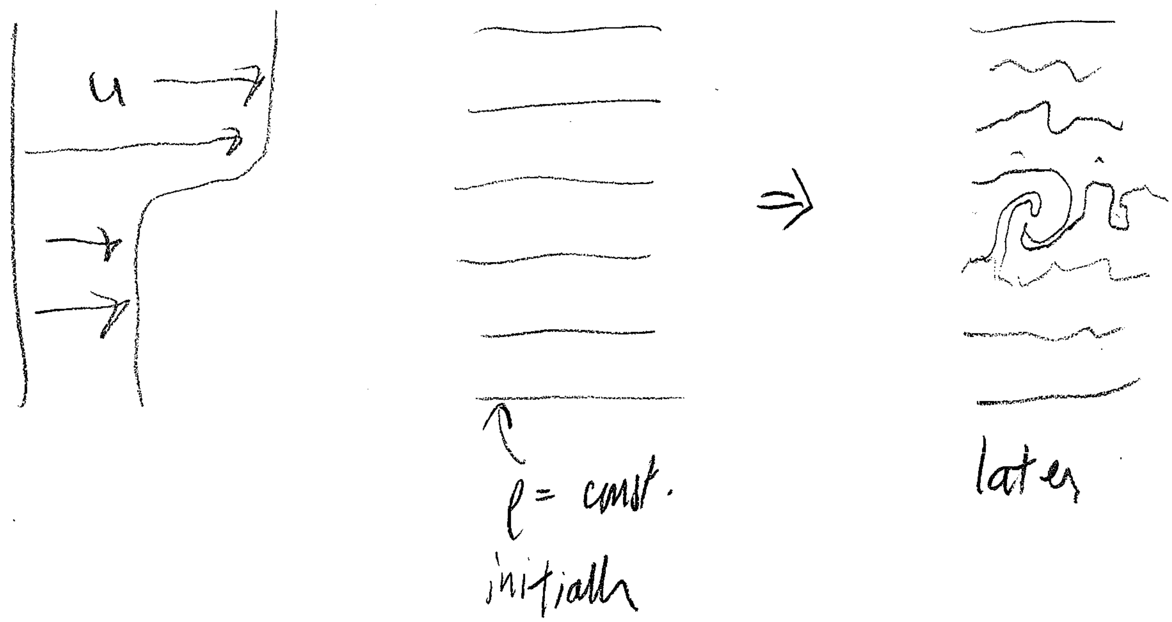




pressure perturbations cause anomaly to grow



Stratified shear flows also unstable sometimes



⇒ turbulence

Kelvin-Helmholtz Instability "K-H"

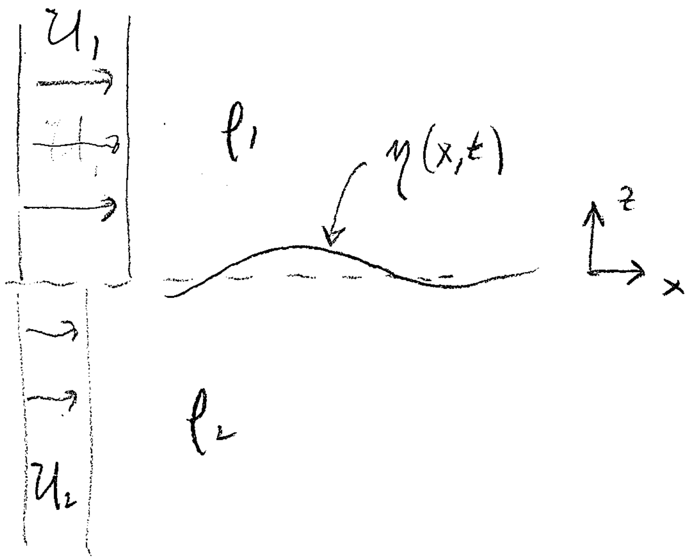
<Start here>

Consider 2-layer flow

irrotational

inviscid

barotropic in each layer



- can use methods of surface gravity waves in each layer and match pressure at interface

eg. in Layer 1

$$\underline{u}_1 = (U_1 + u_1 , w_1) = \nabla \bar{\varphi}_1$$

small wave perturbations $\ll U_1$

$$\Rightarrow \quad \bar{\varphi}_1 = U_1 x + \varphi_1(x,z,t)$$

and $\nabla^2 \bar{\varphi}_1 = \nabla^2 \varphi_1 = 0$

similar for layer 2

$\boxed{\text{KBC } \infty}$ $\varphi_1 \to 0$ as $z \to \infty$, $\varphi_2 \to 0$ as $z \to -\infty$

$\boxed{\text{KBC-I}}$ $\underbrace{w_1 = \frac{D\eta}{Dt} \quad \text{at } z = \eta}_{\text{linearize}}$

interface

$$\Rightarrow \quad \left.\begin{aligned} \frac{\partial \varphi_1}{\partial z} &= \frac{\partial \eta}{\partial t} + u_1 \frac{\partial \eta}{\partial x} \\ \text{and} \quad \frac{\partial \varphi_2}{\partial z} &= \frac{\partial \eta}{\partial t} + u_2 \frac{\partial \eta}{\partial x} \end{aligned}\right\} \text{ at } z = 0$$

Dynamic BC: use unsteady Bernoulli in each layer at $z = \eta$

const. in space

$$\left[\frac{\partial \bar{\varphi}_1}{\partial t} + \frac{1}{2}\left(\bar{\underset{\sim}{u}}_1 \cdot \bar{\underset{\sim}{u}}_1\right) + \frac{\bar{p}_1}{\rho_1} + gz = F_1(t) \right]_{z=\eta}$$

define $\bar{p}_1 = P_1(z) + p_1(x,z,t)$

perturbation

+ similar for layer 2

Note, for undisturbed state

$$\left.\begin{aligned} \frac{1}{2} U_1^2 + \frac{P_1}{\rho_1} &= F_1 \\ \text{and} \quad \frac{1}{2} U_2^2 + \frac{P_2}{\rho_2} &= F_2 \end{aligned}\right\} \text{ and } P_1 = P_2 \text{ at } z = 0 \quad \text{Ⓐ}$$

The linearized form of the full BC is

$$\left[\frac{\partial \varphi_1}{\partial t} + \frac{1}{2} U_1^2 + U_1 \overset{\partial \varphi_1/\partial x}{u_1} + \frac{P_1}{\rho_1} + \frac{p_1}{\rho_1} + g\eta = F_1 \right]_{z=0}$$

and similar for layer 2

then equating the pressure at $z = 0$

+ removing the background Ⓐ give

$$\boxed{\rho_1 \left(\frac{\partial \varphi_1}{\partial t} + U_1 \frac{\partial \varphi_1}{\partial x} + g\eta \right) = \rho_2 \left(\frac{\partial \varphi_2}{\partial t} + U_2 \frac{\partial \varphi_2}{\partial x} + g\eta \right) \quad \text{at } z = 0}$$

DBC-I

Then we search for solutions of the form

e.g. $$\varphi_1 = \Phi_1(z)\,\underbrace{e^{ik(x-ct)}}_{\text{it is implicit that we take the real part}}$$

Note: if c is complex $c = c_R + i c_I$

then $$\mathrm{Re}\left\{ e^{ik(x-ct)} \right\} = \mathrm{Re}\left\{ e^{kc_I t} e^{ik(x-c_R t)} \right\}$$

$$= \underbrace{e^{kc_I t}}_{\text{A growing mode for } c_I > 0} \underbrace{\cos k(x - c_R t)}_{\text{the usual phase propagation with } c_p = c_R}$$

$\Rightarrow$ <u>Instability</u>

continuing with solution, using

$$\eta = \eta_0 e^{ik(x-ct)}$$

then using $\nabla^2 \varphi_1 = 0$

$$\Rightarrow \quad -k^2 \bar{\Phi}_1 + \bar{\Phi}_{1zz} = 0$$

unknown constant

applying $\boxed{\text{KBC}-\infty} \Rightarrow \bar{\Phi}_1 = A e^{-kz}$

and $\boxed{\text{KBC}-\text{I}}$ $\left.\dfrac{\partial \varphi_1}{\partial z} = \dfrac{\partial \eta}{\partial t} + U_1 \dfrac{\partial \eta}{\partial x} \right]_{z=0}$

$$\Rightarrow \quad A = -i(U_1 - c)\eta_0$$

so $\boxed{\varphi_1 = -i(U_1 - c)\eta_0 \, e^{-kz} e^{ik(x-ct)}}$

and similarly $\boxed{\varphi_2 = i(U_2 - c)\eta_0 \, e^{kz} e^{ik(x-ct)}}$

Plugging these into $\boxed{DBC-I}$ yields

$$\rho_1 k(U_1 - c)^2 + \rho_1 g = -\rho_2 k(U_2 - c)^2 + \rho_2 g$$

and this quadratic may be solved for c

$$\boxed{c = \frac{\rho_2 U_2 + \rho_1 U_1}{\rho_2 + \rho_1} \pm \left[\frac{g}{k}\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) - \rho_1 \rho_2 \left(\frac{U_1 - U_2}{\rho_2 + \rho_1}\right)^2 \right]^{\frac{1}{2}}}$$

simplifying this for $U_1 = U_0 + \frac{\Delta U}{2}$, $U_2 = U_0 - \frac{\Delta U}{2}$

$\rho_1 = \rho_0 - \frac{\Delta\rho}{2}$, $\rho_2 = \rho_0 + \frac{\Delta\rho}{2}$, $\Delta\rho \ll \rho_0$

$$\Rightarrow c = U_0 \pm \left[\frac{g'}{k} - \left(\frac{\Delta U}{2}\right)^2 \right]^{\frac{1}{2}} \quad \text{where } g' = \frac{g\Delta\rho}{2\rho_0} \text{ "reduced gravity"}$$

advection by mean flow

for $\Delta U = 0 \Rightarrow c = \sqrt{\frac{g'}{k}}$, like $\sqrt{g/k}$ but much slower!

first imaginary root when $\frac{g'}{k} = \left(\frac{\Delta U}{2}\right)^2 \Rightarrow$ growing instability

$$\Rightarrow k_{CRIT} = \frac{g'}{(\Delta U/2)^2}$$

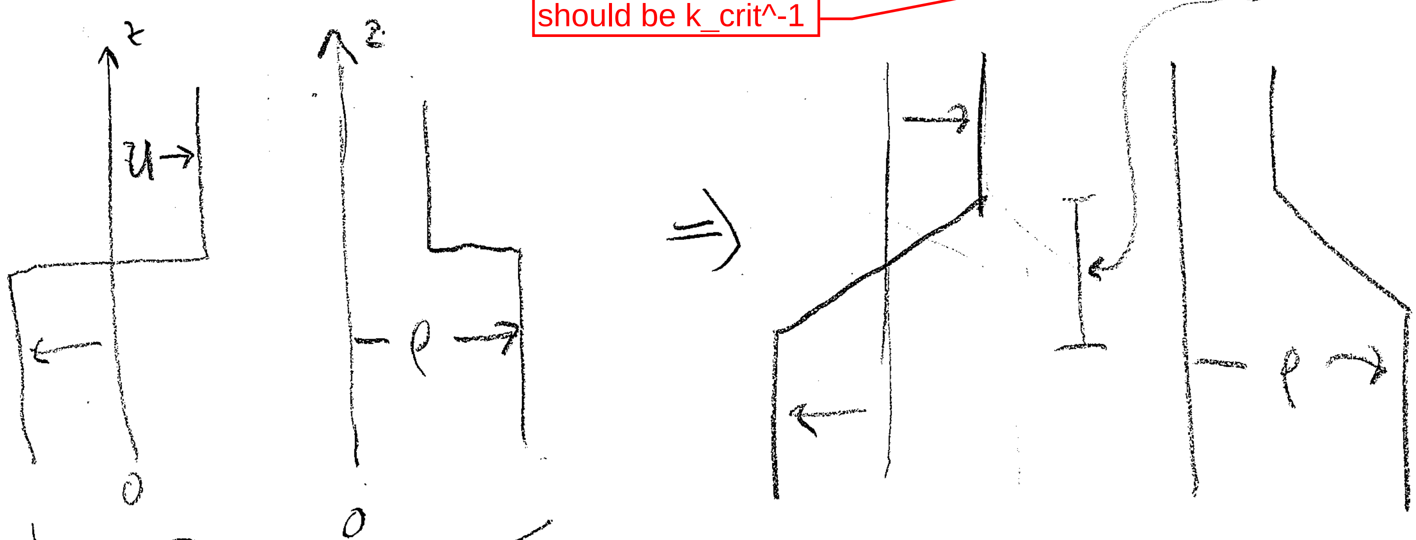
$\Rightarrow$ There is always an unstable wavelength (for $|\Delta u| > 0$)

Result for instability of stratified shear flows with $u = u(z)$ and $\rho = \rho(z)$ is that they are unstable for

$$R_i = \text{"Richardson \#"} = \frac{-\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}}{\left(\frac{\partial u}{\partial z}\right)^2} < \frac{1}{4} \quad (*)$$

In the K-H problem the interface will mix until $(*)$ is satisfied, a thickness of about k^{-1}.

should be k_crit^-1



$t = 0$ $\Rightarrow$ $t =$ after mixing