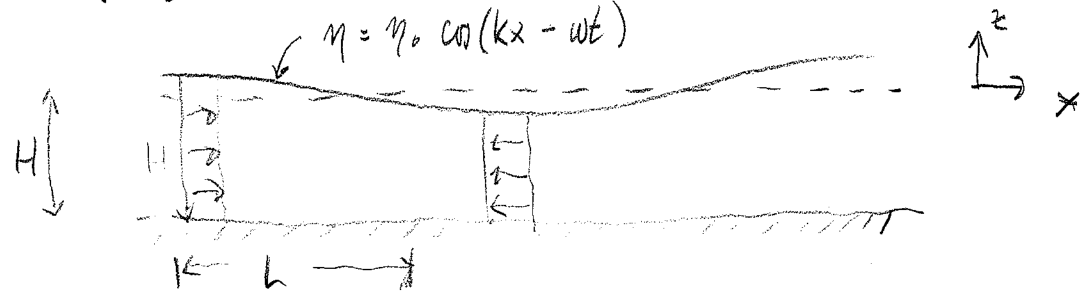
DEEP WATER WAVES

Recall Shallow Water Waves:



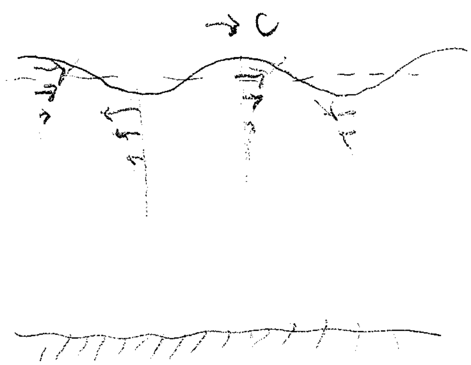
pattern moves at phase speed $c_p = \frac{\omega}{k} = \sqrt{gH}$ → to the right →

Hydrostatic because $\frac{H}{L} \ll 1 \Rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\eta_x$ in $\boxed{\text{xmom}}$

and so $\frac{\partial u}{\partial z} = 0$ (note $[u] \ll c_p$)

What if $\frac{H}{L}$ is not $\ll 1$? ⇒ non-hydrostatic

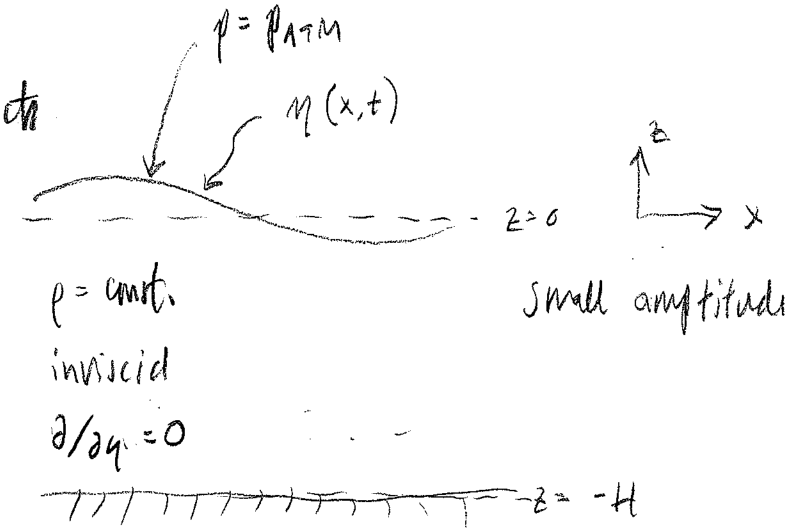
Deep Water Waves



We find:

- $u = u(z)$
- $c_p < \sqrt{gH}$
- Waves are "dispersive"
  ⇒ speed depends on wavelength
  (shorter λ ⇒ slower)

Physical Aspects



Small amplitude $[\eta] \ll H$

irrotational at $t=0$ $\Rightarrow$ irrotational forever (KCT)

$$\Rightarrow \underset{\sim}{u} = \nabla \varphi \qquad (u = \varphi_x,\ w = \varphi_z)$$

also incompressible $\Rightarrow$ $\nabla^2 \varphi = 0$ Laplace eqn. $(\varphi_{xx} + \varphi_{zz} = 0)$

<u>Kinematic boundary conditions</u>

Bottom: $w(z=-H) = 0 \Rightarrow \boxed{\dfrac{\partial \varphi}{\partial z}(x,-H,t) = 0}$ KBC-B

Surface: a fluid parcel on the surface stays on the surface

$$\Rightarrow \frac{D\eta}{Dt} = \underbrace{w(x,\eta)}_{\text{Taylor series expansion}} \qquad \text{KBC-S}$$

$$\eta_t + \cancel{u\eta_x} \doteq w(x,0,t) + \cancel{\eta \frac{\partial w}{\partial z}(x,0,t)}$$

drop for $uk \ll w$ or $\frac{u}{c} \ll 1$

drop if vertical penetration of wave signal is $\gg \eta$

so $\boxed{\varphi_z(x,0,t) = \frac{\partial \eta}{\partial t}}$ KBC-S

Dynamic Boundary Condition

$\underset{\sim}{x}$ mom

$$\underbrace{\underset{\sim}{u}_t}_{\frac{\partial}{\partial t}(\nabla\varphi) = \nabla(\varphi_t)} + \nabla\left(\tfrac{1}{2}\,\underset{\sim}{u}\cdot\underset{\sim}{u}\right) + \cancel{\underset{\sim}{\omega}}\times\underset{\sim}{u} = -\frac{1}{\rho}\nabla p - \nabla(gz)$$

0 irrotational

$$\Rightarrow \quad \nabla\left(\varphi_t + \tfrac{1}{2}\,\underset{\sim}{u}\cdot\underset{\sim}{u} + \frac{p}{\rho} + gz\right) = 0$$

$\Rightarrow$ Generalized Bernoulli Theorem: (start here)

$$\varphi_t + \tfrac{1}{2}\,\cancel{\underset{\sim}{u}}\cdot\underset{\sim}{u} + \frac{p}{\rho} + gz = F(t)$$

some constant field only a function of time.

neglect for $\frac{u}{c} \ll 1$

evaluate at the free surface : $p = p_{ATM}$, $z = \eta$

$$\underbrace{\frac{\partial}{\partial t}\varphi(x,\eta,t)}_{\frac{\partial}{\partial t}\left[\varphi(x,0,t) + \eta\frac{\partial}{\partial t}\varphi(x,0,t)\right]} = -g\eta + \left(F - \frac{p_{ATM}}{\rho}\right)$$

neglect

Note: F influences φ but not $\nabla\varphi$ and can be chosen at our convenience $\Rightarrow$ assume $F = p_{ATM}/\rho$

$\Rightarrow$ $\boxed{\varphi_t(x,0,t) = -g\eta}$ DBC-S

Summarizing the math problem

Solve $\nabla^2 \varphi = 0$ subject to

KBC - B $\quad \dfrac{\partial \varphi}{\partial z}(x, -H, t) = 0$

KBC - S $\quad \dfrac{\partial \varphi}{\partial z}(x, 0, t) = \dfrac{\partial \eta}{\partial t}$

DBC - S $\quad \dfrac{\partial \varphi}{\partial t}(x, 0, t) = -g\eta$

Search for wave-like solutions, with $\eta = \eta_0 \cos(kx - \omega t)$

$$\Rightarrow \eta_t = \omega \eta_0 \sin(kx - \omega t)$$

Guess a solution of the form

$$\varphi = \Phi(z) \sin(kx - \omega t)$$

Then $\nabla^2 \varphi = 0 \Rightarrow \Phi_{zz} - k^2 \Phi = 0$

which has general solution

$$\Phi = \Phi_1 e^{kz} + \Phi_2 e^{-kz}$$

constants

Applying KBC - B + KBC - S

after some manipulation you can show

$$\varphi = \underbrace{\Phi_0}_{\text{constant} \, = \, \frac{\eta_0 \omega}{k} \frac{1}{\sinh(kH)}} \cosh\left[k(z+H)\right] \sin(kx - \omega t)$$

But the solution is incomplete because we need to ensure that k and ω satisfy DBC - S

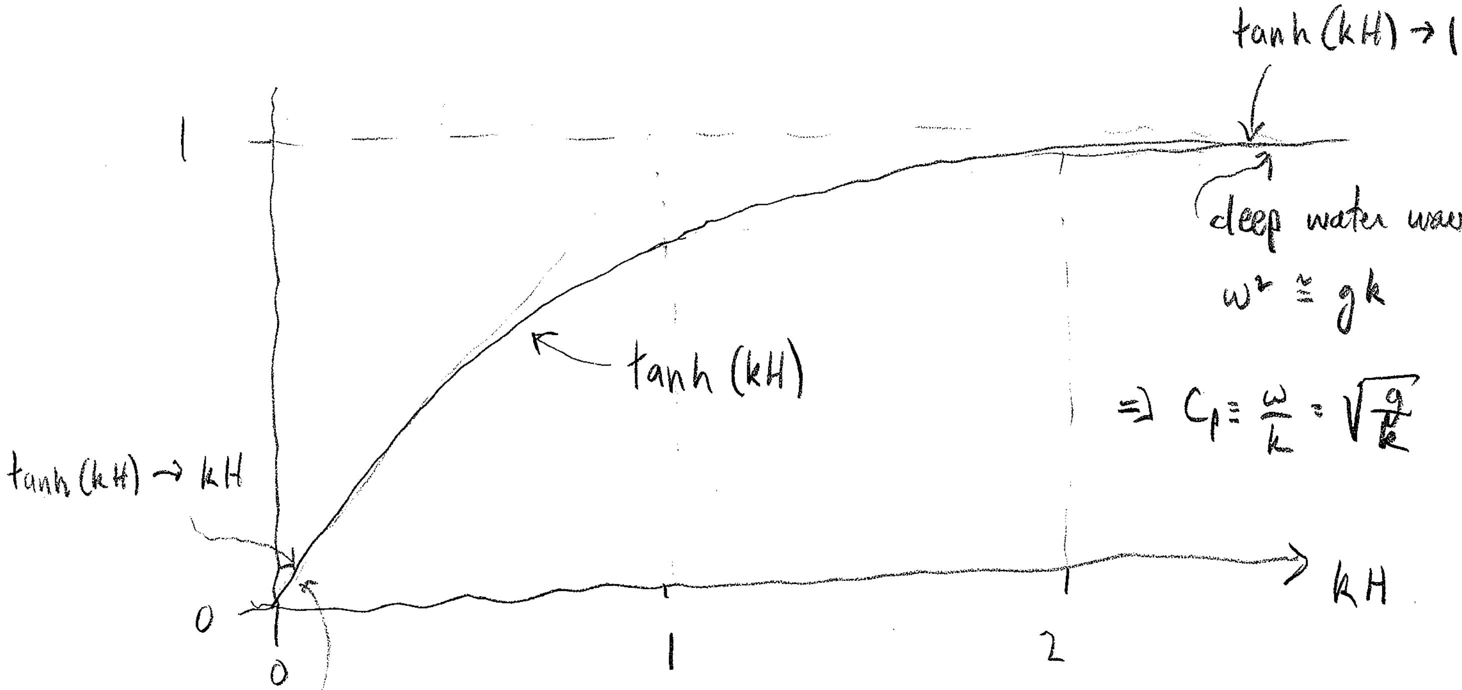
Combine DBC - S + KBC - S and eliminate η to find

$$\left[\varphi_{tt} = -g\,\varphi_z\right]_{z=0}$$

Plugging in our solution for φ gives

$$\boxed{\omega^2 = gk \tanh(kH)}$$

In general $\omega = \omega(k)$ is called the "dispersion relation" (why?)



deep water waves

$\omega^2 \approx g k$

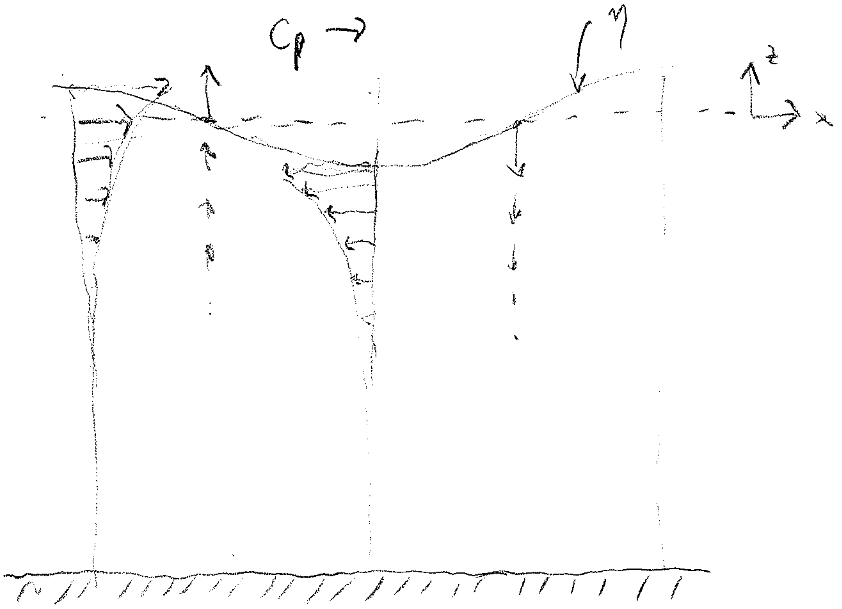
$\Rightarrow C_p \equiv \frac{\omega}{k} = \sqrt{\frac{g}{k}}$

shallow water waves

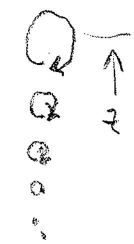
$\omega^2 \approx g k^2 H$

$\Rightarrow C_p \equiv \frac{\omega}{k} = \sqrt{gH}$

Deep Water Waves $H \gg \lambda$ (wavelength)

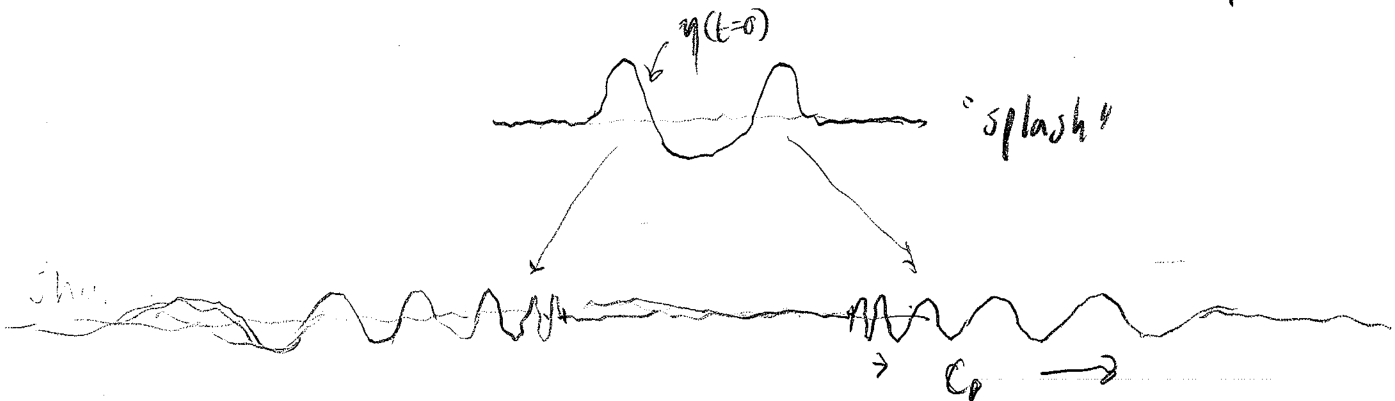


- $u + w$ same magnitude
- Both decay away from the surface as $\exp(kz)$
- parcel paths are circles

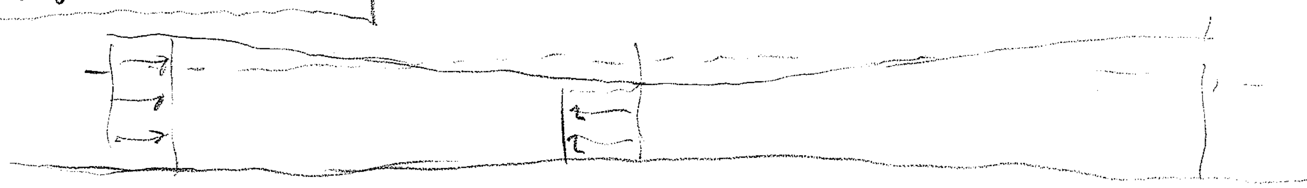


- $c_p = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$

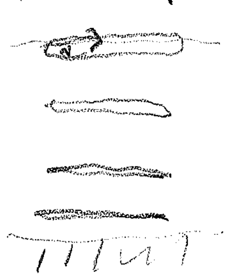
$\Rightarrow$ waves with longer wavelengths travel faster : "dispersive"



Shallow Water Waves $H \ll \lambda$



- $w \ll u$
- $u \sim$ const. with depth
- parcel paths are ~ lines



- $c_p = \sqrt{gH}$

$\Rightarrow$ waves of all wavelengths travel same speed (faster than deep water waves)

"non-dispersive"