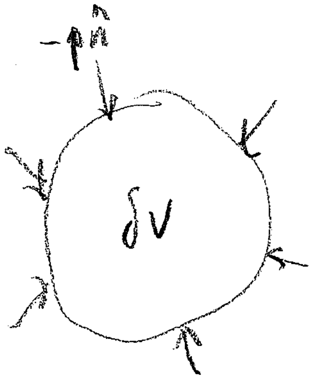
I.6 Gauss Divergence Theorems, buoyancy, and (inviscid) Momentum Conservation

- Recall the Force $\propto$ Pressure Gradient derivation

$$\underset{\sim}{F}^{\text{pressure}} = \lim_{\delta V \to 0} \int_{\delta A} (-p\hat{n})\, dA = -\delta V\, \nabla p = \lim_{\delta V \to 0} \int_{\delta V} (-\nabla p)\, dV$$



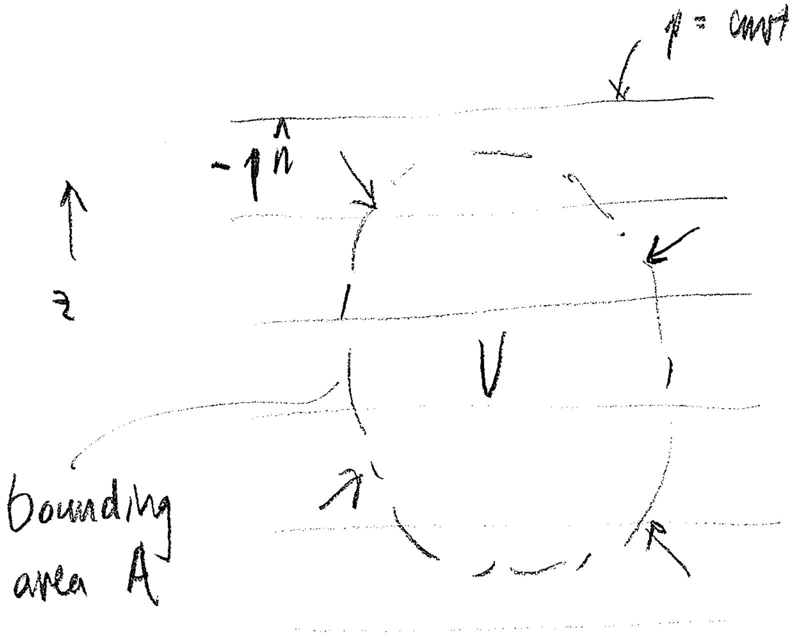
an <u>example</u> of Gauss Divergence Theorem

For any scalar field $\varphi(\underset{\sim}{x})$

$$\int_V \nabla \varphi \, dV = \int_A \varphi \hat{n}\, dA \qquad (*)$$

- valid for <u>any</u> volume
- doesn't have to move w/ fluid

Example : "Buoyancy" ≡ force on a volume due to hydrostatic pressure



$$\underset{\sim}{F}^{\text{pressure}} = -\int_A p\,\hat{n}\,dA$$

$$= -\int_V \nabla p\, dV \quad \text{by } (*)$$

$$= \boxed{\hat{k}\int_V \rho g\, dV = \text{"Buoyancy"}}$$

(using $\frac{\partial p}{\partial z} = -\rho g$)

- for $\rho = \rho_0 =$ const., Buoyancy $= \rho_0 g V$

- also works for $\rho \neq$ const.

Other form of the Gauss Divergence Theorem is:

$$\boxed{\begin{array}{l}\text{for vector field } \underset{\sim}{c}(\underset{\sim}{x}) \\[2ex] \displaystyle\int_V \nabla \cdot \underset{\sim}{c}\, dV = \int_A \underset{\sim}{c}\, \hat{n}\, dA\end{array}}$$

like $\dfrac{D\delta V}{Dt} = \delta V (\nabla \cdot \underset{\sim}{u})$ in $\boxed{\text{mass}}$ derivation

$\boxed{\text{Conservation of Momentum following a fluid parcel}}$

$$\underset{\sim}{F} = m\,\underset{\sim}{a}$$

$m = \delta V \rho$

$\underset{\sim}{a} = \frac{D\underset{\sim}{u}}{Dt}$

$\underset{\sim}{F}$ = gravity + pressure gradient

$-\hat{k} g\, \delta V \rho \qquad -\delta V\, \nabla p \qquad \left(\text{recall } \frac{\underset{\sim}{x} \text{ force}}{\text{unit vol.}} = -\nabla p\right)$

rearranging:

$$\therefore \quad \cancel{\delta V}\,\rho \frac{D\underset{\sim}{u}}{Dt} = -\cancel{\delta V}\,\nabla p - \hat{k} g \rho \cancel{\delta V}$$

or
$$\boxed{\frac{D\underset{\sim}{u}}{Dt} = -\frac{1}{\rho}\nabla p - \hat{k} g} \qquad \boxed{\underset{\sim}{x} \text{ mom}}$$

(inviscid)

"Euler's equation (s)"

Now - this is really <u>three</u> equations

$\boxed{\text{x mom}}$ $$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$\boxed{\text{y mom}}$ $$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$\boxed{\text{z mom}}$ $$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

remember $\frac{\partial p}{\partial z} = -\rho g$ "hydrostatic balance"

this was $\boxed{\text{z mom}}$ with $\frac{Dw}{Dt} = 0$