I.5 Conservation of Mass in a fluid

a fluid parcel conserves its mass (*)

mass $= \rho\, \delta V$ for parcel with volume δV

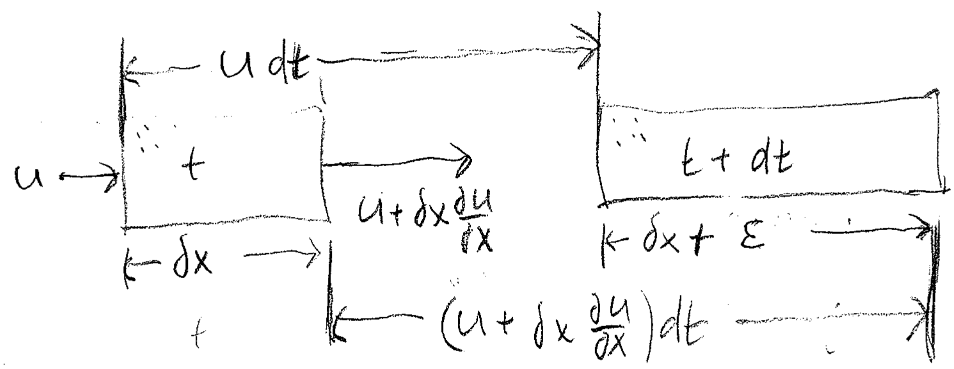
$$\therefore \quad \frac{D(\rho\, \delta V)}{Dt} = 0 \quad \text{is a way to write } (*)$$

$$\Rightarrow \quad \rho \frac{D\, \delta V}{Dt} + \delta V \frac{D\rho}{Dt} = 0$$

or
$$\frac{1}{\rho}\frac{D\rho}{Dt} + \underbrace{\frac{1}{\delta V}\frac{D\, \delta V}{Dt}} = 0 \qquad (**)$$

we can write this in a more useful way

consider a fluid parcel whose volume is changing ($\delta V \uparrow \Rightarrow \rho \downarrow$)



and $\varepsilon = \delta x \frac{\partial u}{\partial x}\, dt$

so $$\frac{D\,\delta x}{Dt} = \frac{\varepsilon}{dt} = \delta x \frac{\partial u}{\partial x} \Rightarrow \boxed{\frac{1}{\delta x}\frac{D\,\delta x}{Dt} = \frac{\partial u}{\partial x}}$$

then note that

$$\frac{1}{\delta V}\frac{D\,\delta V}{Dt} = \frac{1}{\delta x\, \delta y\, \delta z}\frac{D\,\delta x\, \delta y\, \delta z}{Dt} = \frac{1}{\delta x}\frac{D\,\delta x}{Dt} + \frac{1}{\delta y}\frac{D\,\delta y}{Dt} + \frac{1}{\delta z}\frac{D\,\delta z}{Dt}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \boxed{\nabla \cdot \underset{\sim}{u} = \frac{1}{\delta V}\frac{D\,\delta V}{Dt}}$$

"the divergence of $\underset{\sim}{u}$" (scalar field)

(fractional rate of change of volume following a fluid parcel) so now we can rewrite (★★) as

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \underset{\sim}{u} = 0} \quad \boxed{\text{mass}}$$

equation for mass conservation

fractional rate of change of density following a fluid parcel.

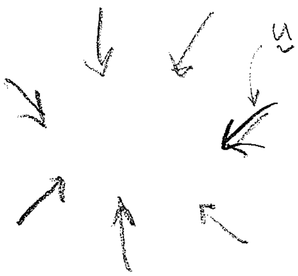
- exact
- a Lagrangian idea, written in Eulerian terms

Often the change in ρ is negligible...

e.g. $$\overbrace{\frac{1}{\rho}\frac{D\rho}{Dt}}^{10^{-3}\,s^{-1}} + \overbrace{\underbrace{u_x}_{3\,s^{-1}} + \underbrace{v_y}_{-1\,s^{-1}} + \underbrace{w_z}_{-1.001\,s^{-1}}}^{-10^{-3}\,s^{-1}} = 0$$

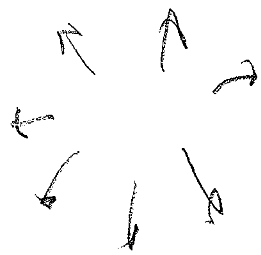
Then we approximate $\boxed{\text{mass}}$ as

$$\boxed{\nabla \cdot \underset{\sim}{u} = 0} \quad \text{"incompressible"}$$



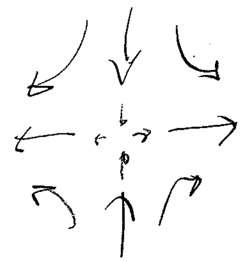
$\nabla \cdot \underset{\sim}{u} < 0$

convergence



$\nabla \cdot \underset{\sim}{u} > 0$

divergence



$\nabla \cdot \underset{\sim}{u} = 0$

incompressible

compressible