# Velocity, Eulerian & Lagrangian points of view, the Material Derivative

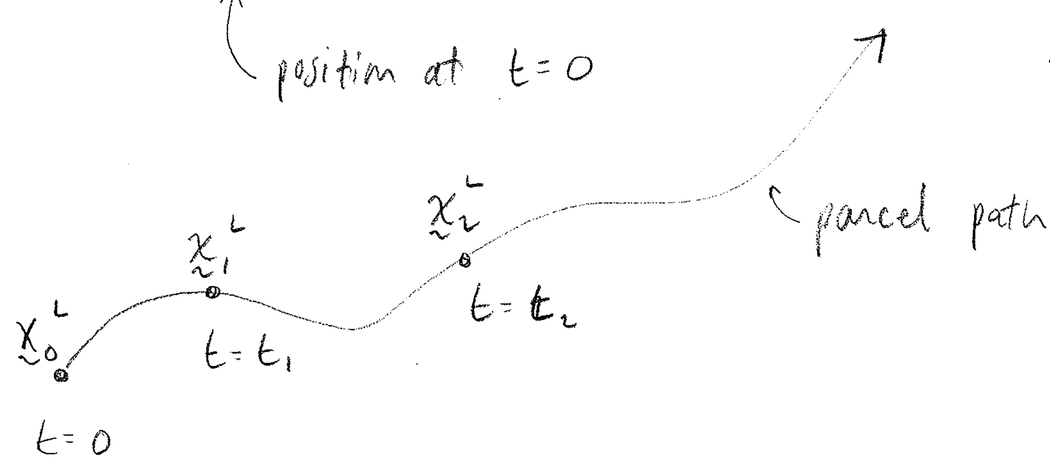
Motivation: conservation laws (like $\underset{\sim}{F} = m\underset{\sim}{a}$) most naturally apply following a fluid parcel [defined as a "Lagrangian" coordinate system]. But measurements and mathematical tools describe changes at fixed points in space [defined as an "Eulerian" coordinate system].

Define the position of a fluid parcel

L = Lagrangian

$$\underset{\sim}{x}^L = \underset{\sim}{x}^L(\underset{\sim}{x}_0^L, t)$$

position at $t=0$





$\underset{\sim}{x}_0^L$ is a way of labeling a parcel

Velocity is the rate of change of position following a fluid parcel

$$\text{velocity} \equiv \left.\frac{\partial \underset{\sim}{x}^L}{\partial t}\right|_{\underset{\sim}{x}^L_0 = \text{const.}} = \left.\underset{\sim}{u}\right|_{\underset{\sim}{x}^L_0 = \text{const}}$$

and if we consider all possible $\underset{\sim}{x}^L_0$ then $\underset{\sim}{u}$ spans all $\underset{\sim}{x}$ and $t$ of the fluid system, and we have the

velocity field $\underset{\sim}{u}(\underset{\sim}{x}, t)$ $[m\, s^{-1}]$ vector field

$$= (u, v, w) = \hat{i}\, u(\underset{\sim}{x}, t) + \hat{j}\, v(\underset{\sim}{x}, t) + \hat{k}\, w(\underset{\sim}{x}, t)$$

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Now consider small changes in the value of some scalar field $T(\underset{\sim}{x}, t)$, using the chain rule

$$\delta T = \frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z$$

where $\delta \underset{\sim}{x}$ is an arbitrary change in position dividing by $\delta t$

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial T}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial T}{\partial z}\frac{\delta z}{\delta t}$$

then assume $\delta \underset{\sim}{x}$ is not arbitrary, but is following a fluid parcel, so $\frac{\delta \underset{\sim}{x}}{\delta t} = \underset{\sim}{u}$ $\quad \left( \frac{\delta x}{\delta t} = u, \text{ etc} \ldots \right)$

In this special case we use notation $\frac{\delta T}{\delta t} = \frac{Dt}{Dt}$

and

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

$= T_t + \underset{\sim}{u} \cdot \nabla T$ ⇐ Dot product $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$= T_t + u_i \frac{\partial T}{\partial x_i}$ ⇐ Indicial notation: sum on repeated indices

In words $\frac{DT}{Dt}$ = "the rate of change of T following a fluid parcel"

And $\boxed{\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + \overbrace{\underset{\sim}{u} \cdot \nabla(\,)}^{(*)}}$ is the "Material Derivative"

NOTE: $\frac{\partial}{\partial x}$ and etc in (*) are Eulerian, meaning in coordinates fixed in space.