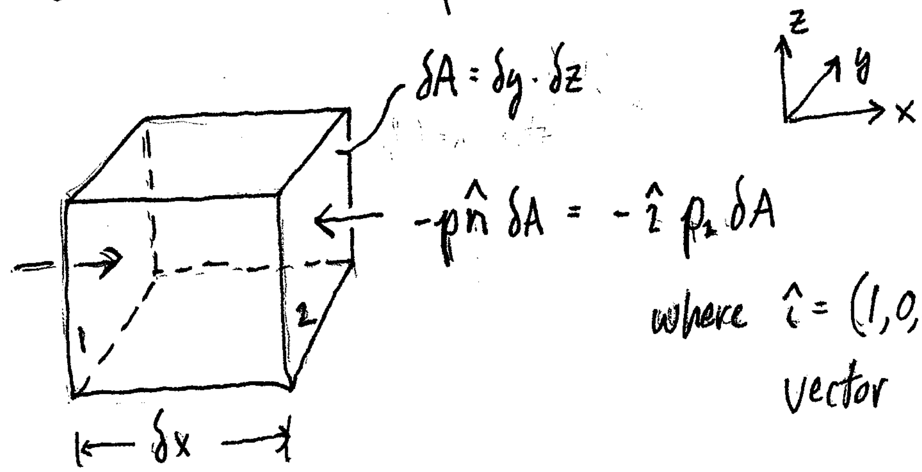
Simpler case: make the parcel a cube of volume $\delta V$

$\delta A = \delta y \cdot \delta z$




$-p\hat{n}\,\delta A = p_1 \hat{\imath}\,\delta A$

$-p\hat{n}\,\delta A = -\hat{\imath}\,p_2\,\delta A$

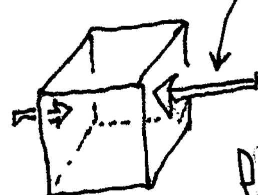
where $\hat{\imath} = (1,0,0)$ unit vector in x-direction

$$\text{Net x-force} = -(p_2 - p_1)\,\delta A = -(p_2 - p_1)\,\delta y\,\delta z = -\frac{(p_2 - p_1)}{\delta x}\underbrace{\delta x\,\delta y\,\delta z}_{\delta V}$$

or

$$\frac{\text{x-force}}{\text{unit volume}} = \lim_{\delta V \to 0} \frac{\text{net x-force}}{\delta V} = \lim_{\delta x \to 0} \frac{-(p_2 - p_1)}{\delta x} = -\frac{\partial p}{\partial x}$$

higher pressure on this side ($\partial p/\partial x > 0$)



pushes cube to the left

Same argument for y- and z-directions gives ...

$$\Rightarrow \frac{\underset{\sim}{\text{x}}\text{-force}}{\text{unit volume}} = -\hat{\imath}\frac{\partial p}{\partial x} - \underbrace{\hat{\jmath}}_{(0,1,0)}\frac{\partial p}{\partial y} - \underbrace{\hat{k}}_{(0,0,1)}\frac{\partial p}{\partial z} = -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$

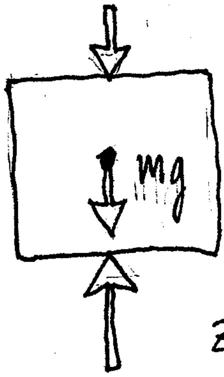
$$= -(p_x, p_y, p_z) \quad \text{subscript notation}$$

$$= -\nabla p \quad \text{where } \nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \text{"grad" the gradient operator}$$

# Hydrostatic balance: pressure field balances gravity

parcel has mass:

$m = \rho * \delta V$




$$\frac{\text{z-force from pressure}}{\text{unit volume}} = -\hat{k}\frac{\partial p}{\partial z}$$

$$\frac{\text{z-force from gravity}}{\text{unit vol.}} = \frac{-\hat{k}\,m\,\overset{9.8\ \mathrm{m\,s^{-2}}}{g}}{\text{unit vol}} = -\hat{k}\rho g$$

$$\frac{\text{total z-force}}{\text{unit vol.}} = 0 = -\hat{k}\frac{\partial p}{\partial z} - \hat{k}\rho g$$

implies no acceleration of the parcel

or

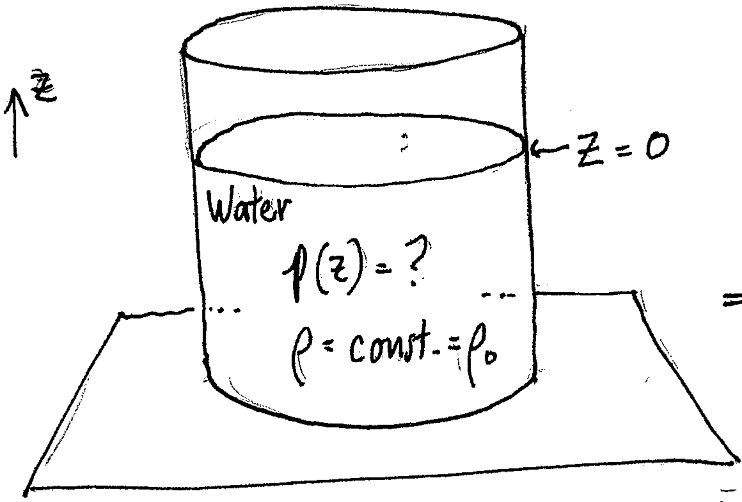
$$\boxed{\frac{\partial p}{\partial z} = -\rho g}$$

The "hydrostatic balance"

- simplest momentum equation
- works even when $\rho \neq$ const.
- excellent approximation for 95% of Atm. + Ocn. flows!

Example: pressure field in a tank of water

$p = p_{ATM}$ in the air



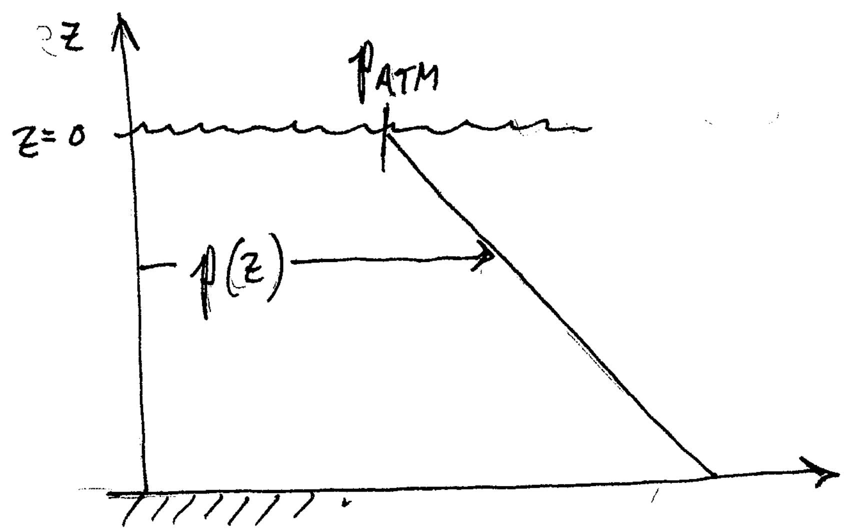

$$\int_z^0 \left[ \frac{\partial p}{\partial z'} = -\rho g \right] dz'$$

$$\Rightarrow p(0) - p(z) = -\rho g (0 - z)$$

$\uparrow$ $p_{ATM}$

$$\Rightarrow p(z) = p_{ATM} - \rho_0 g z$$

Note: $z$ is <u>negative</u> in the water




Note: $p_{ATM} \cong 10^5$ Pa = 1 "bar" = weight of 10,000 kg of air / $m^2$ !
(at sea level)

and for water $\rho_0 \cong 1000$ kg $m^{-3}$

$\Rightarrow$ $p \to 2 \times p_{ATM}$ at 10 m depth