$\boxed{\text{Vorticity Equation}}$

- created by forming $\nabla \times \boxed{\underset{\sim}{x} \text{ mom}}$

- assume ρ = const., so $\nabla \cdot \underset{\sim}{u} = 0$ incompressible

- Note: $\nabla \cdot [\nabla \times (\;)] = 0 \Rightarrow \nabla \cdot \underset{\sim}{\omega} = 0$ (+)

- Note: $\nabla \times [\nabla (\;)] = 0$ (*)

$$\nabla \times \left[\underset{①}{\frac{\partial \underset{\sim}{u}}{\partial t}} + \underset{②}{\underset{\sim}{u} \cdot \nabla \underset{\sim}{u}} = \underset{③}{-\frac{1}{\rho} \nabla p} - \underset{④}{\nabla (gz)} + \underset{⑤}{\nu \nabla^2 \underset{\sim}{u}} \right]$$

Term by term

① $\nabla \times \frac{\partial \underset{\sim}{u}}{\partial t} = \frac{\partial \underset{\sim}{\omega}}{\partial t}$

② $\nabla \times (\underset{\sim}{u} \cdot \nabla \underset{\sim}{u}) = \nabla \times \left[\underbrace{\cancel{\nabla (\tfrac{1}{2} \underset{\sim}{u} \cdot \underset{\sim}{u})}}_{=0 \text{ by } (*)} + \underset{\sim}{\omega} \times \underset{\sim}{u} \right]$

$= \nabla \times \underset{\sim}{\omega} \times \underset{\sim}{u}$ + use $\nabla \times \underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{A}(\nabla \cdot \underset{\sim}{B}) + (\underset{\sim}{B} \cdot \nabla)\underset{\sim}{A} - \underset{\sim}{B}(\nabla \cdot \underset{\sim}{A}) - (\underset{\sim}{A} \cdot \nabla)\underset{\sim}{B}$

$= \underbrace{\underset{\sim}{\omega}(\cancel{\nabla \cdot \underset{\sim}{u}})}_{=0} + (\underset{\sim}{u} \cdot \nabla)\underset{\sim}{\omega} - \underbrace{\underset{\sim}{u}(\cancel{\nabla \cdot \underset{\sim}{\omega}})}_{=0 \text{ by } (+)} - (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{u}$

$= (\underset{\sim}{u} \cdot \nabla)\underset{\sim}{\omega} - \underset{\sim}{\omega} \cdot \nabla \underset{\sim}{u}$

③ + ④ = 0 by (*)

(5) $\nabla \times (\nu \nabla^2 \underset{\sim}{u}) = \nu \nabla^2 \underset{\sim}{\omega}$

So we have

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{\omega} - \underset{\sim}{\omega} \cdot \nabla \underset{\sim}{u} = \nu \nabla^2 \underset{\sim}{\omega}$$

or

$$\boxed{\frac{D\underset{\sim}{\omega}}{Dt} = \underset{\sim}{\omega} \cdot \nabla \underset{\sim}{u} + \nu \nabla^2 \underset{\sim}{\omega}} \quad \boxed{\underset{\sim}{\omega}}$$

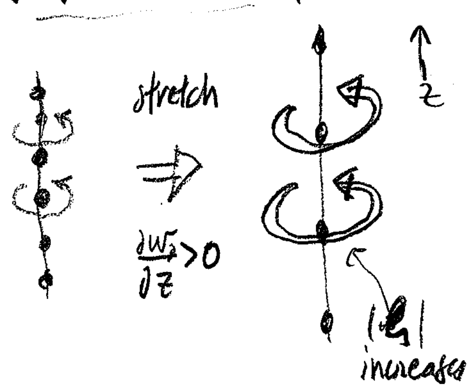
Rate of change of vorticity (vector) following a fluid parcel = stretching or tilting + viscous diffusion of vorticity

looks like a passive tracer except for this term ! ? ?

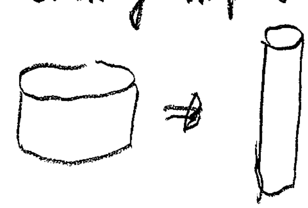
Consider stretching, meaning elongation of a vortex line

for vertical vorticity $\underset{\sim}{\omega} \cdot \nabla \underset{\sim}{u} = \zeta \frac{\partial w}{\partial z}$ where $\zeta = \hat{k} \cdot \underset{\sim}{\omega} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

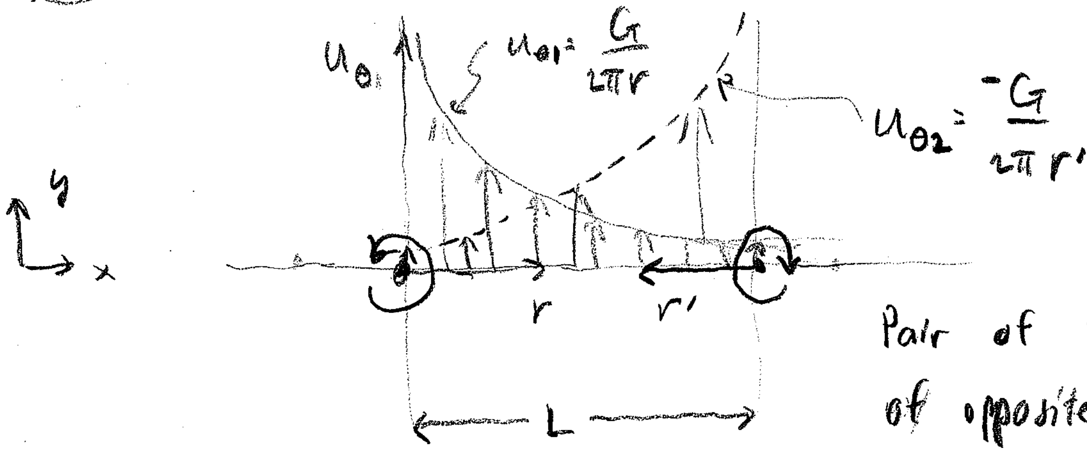
line of fluid parcels (& vortex lines)



Reason: Stretching implies radial inflow

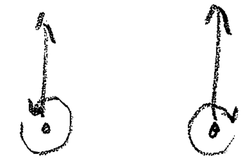


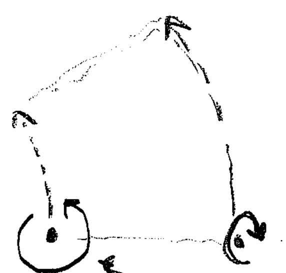
2-D interaction of point vortices



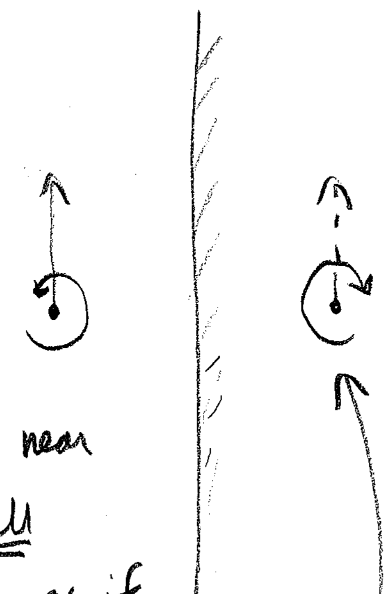
Pair of vorticies of opposite sign

will advect each other with speed $v = \frac{G}{2\pi L}$



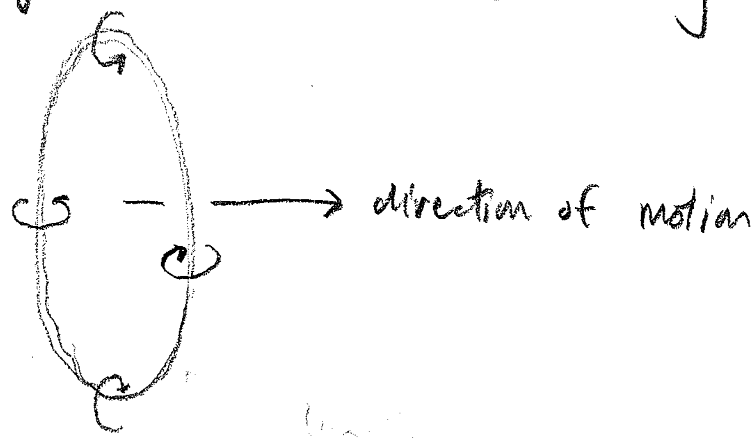


If one is stronger the paths are curved



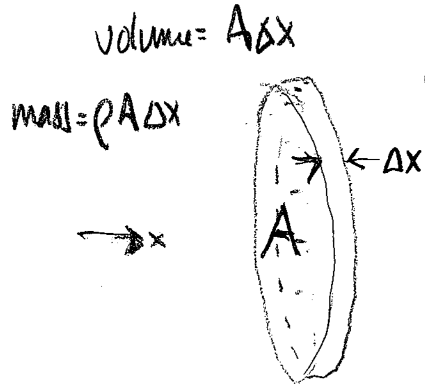
One near a wall moves as if there was an equal and opposite image vortex here

★ Same reasoning works for a <u>vortex ring</u>



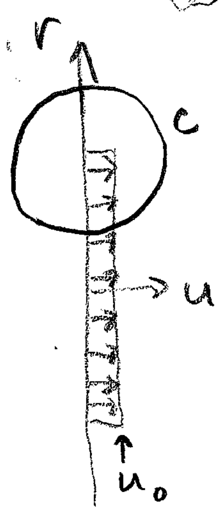
This is an expression of <u>momentum</u>

– Consider creating a vortex ring



(i) apply a body force F to a disk of fluid over time ΔT ("impulse")

$$\Rightarrow \text{momentum} = \rho A \Delta x \int_0^{\Delta T} F\, dt = M$$



(ii) at the start the velocity is like this

– which has circulation $\Gamma = \oint_C \underset{\sim}{u} \cdot \underset{\sim}{dl} = u_0 dx$

and $M = \rho A \underbrace{\Delta x \, u_0}_{\Gamma} \Rightarrow \boxed{\rho A \Gamma = \text{momentum of a vortex ring}}$ (5)

- still approximately correct even as full velocity field develops