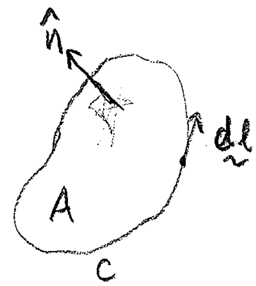
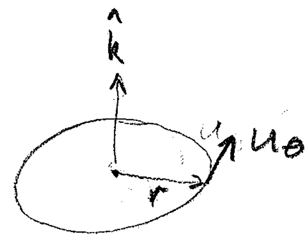
Vortex Dynamics

- fundamental to boundary layers, turbulence, lift, propulsion ("smoke rings"), weather, ocean eddies, Rossby waves



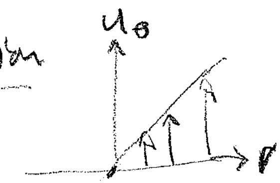
- vorticity $\underset{\sim}{\omega} \equiv \nabla \times \underset{\sim}{u}$
- circulation $\Gamma \equiv \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{l}$
- Stokes Thm. $\Gamma = \int_A \underset{\sim}{\omega} \cdot \hat{n}\, dA$



$\boxed{\text{Example}}$ – circular flow around the origin
– cylindrical polar coordinates

$$\zeta = \hat{k} \cdot \underset{\sim}{\omega} = \frac{1}{r}\frac{d}{dr}(r u_\theta) - \cancel{\frac{1}{r}\frac{\partial u_r}{\partial \theta}} \quad \text{ignore}$$

Solid body Rotation

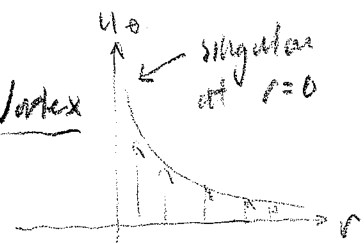


$u_\theta = \Omega r$

$\zeta = \frac{1}{r}\frac{d}{dr}(\Omega r^2) = 2\Omega$ (const.)

$\Gamma = 2\pi \Omega r^2$ (increases w/ r)

Point ("Irrotational") Vortex

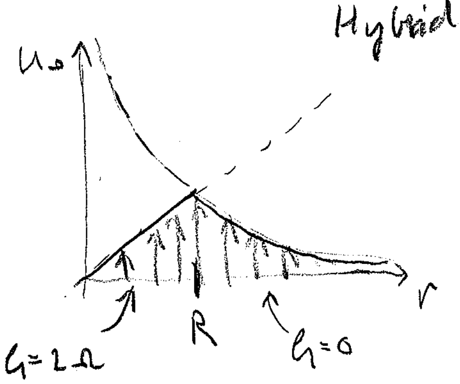


$u_\theta = \frac{G}{2\pi r}$

$\zeta = \frac{1}{r}\frac{d}{dr}\left(\frac{G}{2\pi}\right) = 0$ (irrotational!)

$\Gamma = 2\pi r \frac{G}{2\pi r} = G$ (constant and <u>non zero</u>)

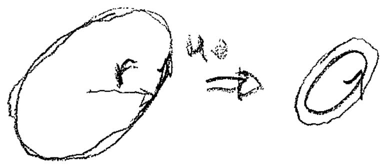
Hybrid: Rankine Vortex (close to <u>real</u> vortices)



- solid body core
- irrotational outside

$$\Gamma = 2\pi \Omega R^2 \quad \text{(constant)}$$

- What happens to Γ for a fluid ring that contracts?



$$\Gamma = 2\pi r u_\theta$$

For a point mass, conservation of angular momentum says $m u_\theta r = \text{const.}$

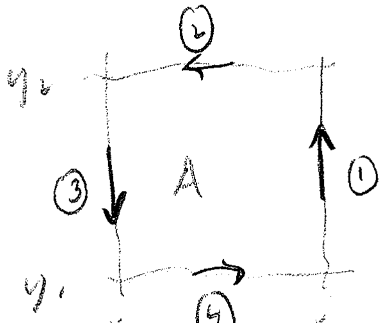
For the ring, $m = \text{const.}$ $\Rightarrow$ $r u_\theta = \text{const.}$

so as r decreases, u_θ increases

$\boxed{\text{and } \Gamma \text{ remains constant.}}$

Note 11/13/2009 A simple example of Stokes' Thm.

2D flow in x, y plane, rectangular area



$$\int_A \underset{\sim}{\omega} \cdot \hat{n}\, dA = \int_A v_x - u_y \, dA$$

$$= \int_{y_1}^{y_2} \Big(\underset{①}{v|_{x_2}} - \underset{③}{v|_{x_1}} \Big) dy - \int_{x_1}^{x_2} \Big(\underset{②}{u|_{y_2}} - \underset{④}{u|_{y_1}} \Big) dx$$

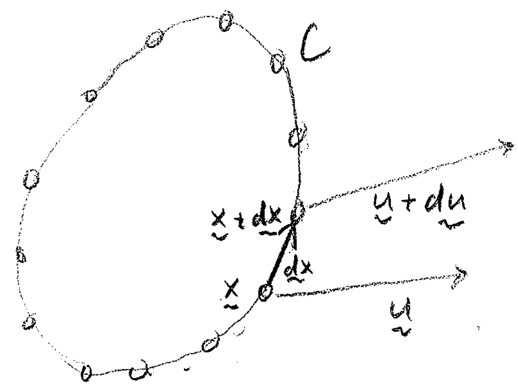
$$= \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{l}$$

Kelvin Circulation Theorem "KCT"

For flow that is inviscid, barotropic (e.g. $\rho = $ const.), has conservative body forces (e.g. gravity):

$$\boxed{\frac{D\Gamma}{Dt} = 0}$$

Proof: consider a fluid ring, a "necklace" of fluid parcels:



$$\Gamma = \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{x}$$

$\underset{\sim}{x}$ gives parcel positions (like x^i)

Note $\dfrac{D\underset{\sim}{x}}{Dt} = \underset{\sim}{u}$ & $\dfrac{D(d\underset{\sim}{x})}{Dt} = d\underset{\sim}{u}$

And $\dfrac{D}{Dt}(\underset{\sim}{u} \cdot d\underset{\sim}{x}) = \underset{\sim}{u} \cdot \dfrac{D(d\underset{\sim}{x})}{Dt} + \dfrac{D\underset{\sim}{u}}{Dt} \cdot d\underset{\sim}{x}$ (*)

(**) Tricky step!

Now $$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\left(\oint_C \underset{\sim}{u} \cdot d\underset{\sim}{x}\right) \overset{(**)}{=} \oint_C \frac{D(\underset{\sim}{u} \cdot d\underset{\sim}{x})}{Dt} = \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{u} + \oint_C \frac{D\underset{\sim}{u}}{Dt} \cdot d\underset{\sim}{x}$$

$$= \underbrace{\cancel{\oint_C d\left(\tfrac{1}{2}\underset{\sim}{u} \cdot \underset{\sim}{u}\right)}}_{0} + \oint_C \frac{D\underset{\sim}{u}}{Dt} \cdot d\underset{\sim}{x}$$

$\rho = $ const $= -\hat{k}g$ inviscid

and $\boxed{\underset{\sim}{x} \text{ mom}}$ gives $\dfrac{D\underset{\sim}{u}}{Dt} = -\dfrac{1}{\rho}\nabla p - \nabla(gz)$

Then since $\oint_C \nabla(\;)\cdot d\underset{\sim}{x} = 0$

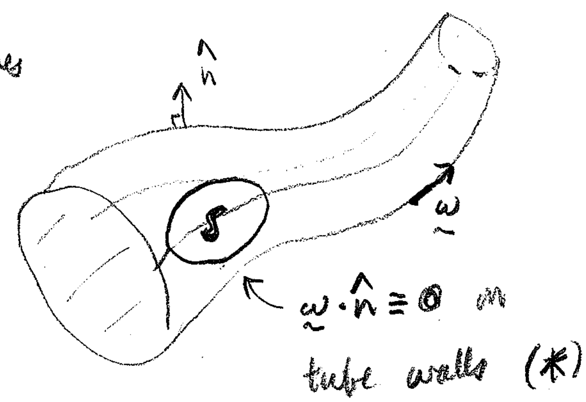
$$\Rightarrow \oint_C \frac{D\underset{\sim}{u}}{Dt}\cdot d\underset{\sim}{x} = 0 \qquad \text{(barotropic, inviscid, conservative body force)}$$

$$\therefore \boxed{\frac{D\Gamma}{Dt} = 0} \Rightarrow \text{irrotational fluid remains irrotational}$$

Concepts: <u>Vortex line</u>: line $\parallel$ to $\underset{\sim}{\omega}$ (like a streamline)

<u>vortex tube</u>: tube made of vortex lines

(like a stream tube)



<u>Helmholtz Vortex Theorems (1)-(4)</u>
(same conditions as for KCT)

(1) Vortex tube moves w/ the fluid

consider patch "S" on the tube

$$\Gamma = \int_S \underset{\sim}{\omega}\cdot\hat{n}\, dA = 0 \quad \text{because of } (*)$$

$$\text{KCT} \Rightarrow \frac{D\Gamma}{Dt} = 0$$

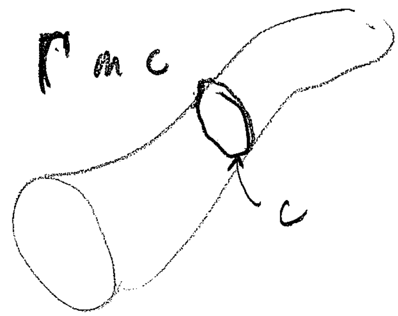
$\therefore$ even as tube deforms the condition $\underset{\sim}{\omega}\cdot\hat{n} = 0$ will be satisfied on the tube walls, so it is still a vortex tube.

(2) Vortex lines move with fluid

proof: just take the limit of a skinny tube!

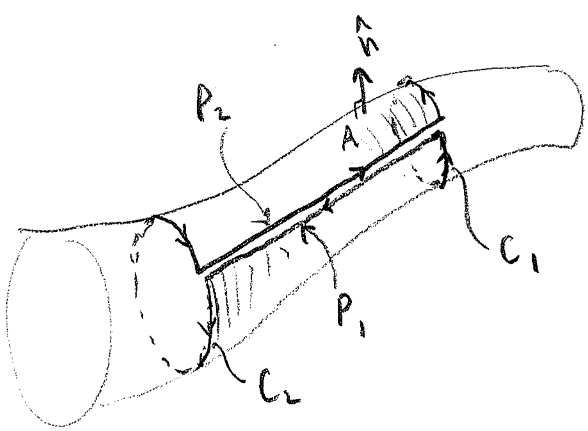
(3) Define vortex tube "strength" as Γ on C

(any path on the tube that encircles it)



Then vortex tube strength is constant along the tube!

Proof



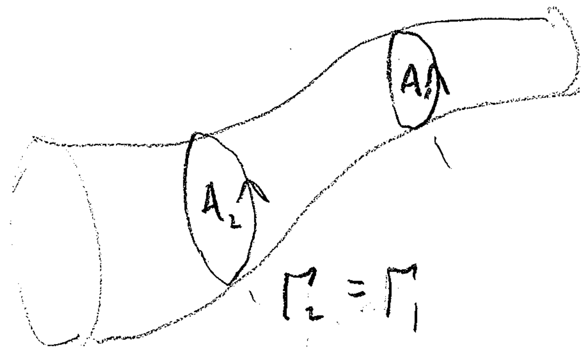
For this path $\int_A \underset{\sim}{\omega} \cdot \hat{n}\, dA = \Gamma_A = 0$ since $\underset{\sim}{\omega} \cdot \hat{n} = 0$

But note that

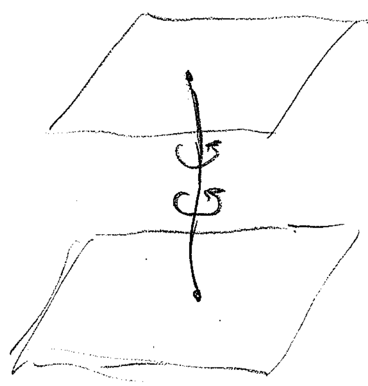
$$\Gamma_A = \underbrace{\int_{C_1} \underset{\sim}{u} \cdot d\underset{\sim}{l}}_{=\Gamma_1} + \underbrace{\int_{P_1} \underset{\sim}{u} \cdot d\underset{\sim}{l} \quad + \quad \int_{P_2} \underset{\sim}{u} \cdot d\underset{\sim}{l}}_{\text{cancel}} + \underbrace{\int_{C_2} \underset{\sim}{u} \cdot d\underset{\sim}{l}}_{=\Gamma_2} = 0$$

$\therefore \Gamma_2 = -\Gamma_1$

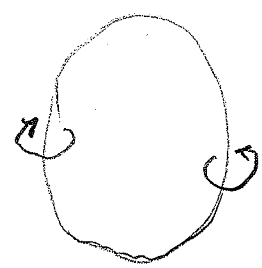
and the two circulations are equal for the same sign of $\oint_C$



(4) Since vortex tube strength is constant along the tube
$\Rightarrow$ vortex tubes (+ lines) cannot end in the fluid



(i) can end at boundaries (solid, or free surface)



(ii) can join to make vortex rings

⟨still inviscid + barotropic⟩