$\boxed{\text{II}.5}$

The Bernoulli Function

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Leonhard Euler 1707-1783 }

A useful integral of $\boxed{\underline{x} \text{ mom}}$

$\boxed{\underline{x} \text{ mom}}$ $$\underbrace{\frac{D\underset{\sim}{u}}{Dt}} = -\frac{1}{\rho}\nabla p - \hat{k} g$$

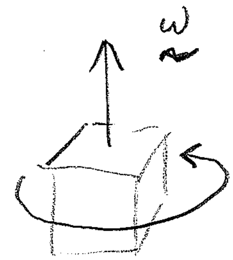
* assume inviscid
* assume ρ = const

can write this as:

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u} = \frac{\partial \underset{\sim}{u}}{\partial t} + \nabla\left(\frac{1}{2}\underset{\sim}{u}\cdot\underset{\sim}{u}\right) + \underset{\sim}{\omega}\times\underset{\sim}{u}$$

where $\underset{\sim}{\omega} = \nabla\times\underset{\sim}{u}$ "vorticity"

twice the rotation rate of a fluid parcel



Note on the cross product:

$$\nabla\times\underset{\sim}{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \hat{i}(w_y - v_z) + \hat{j}(u_z - w_x) + \hat{k}(v_x - u_y)$$

determinant

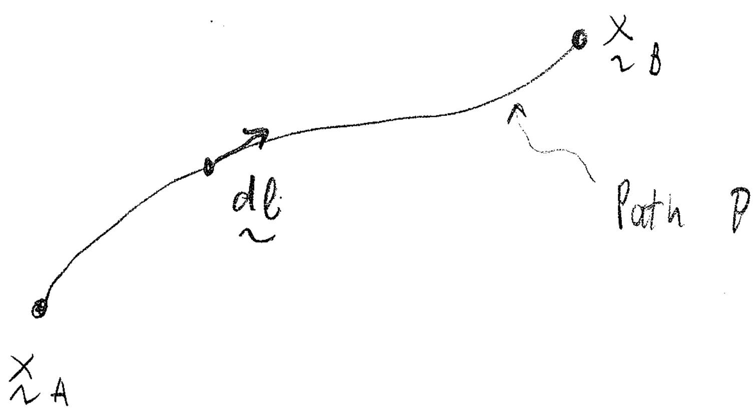
~~A~~ assume steady $\frac{\partial \underset{\sim}{u}}{\partial t} = 0$

note $-\hat{k}g = -\nabla(gz)$

then we may write $\boxed{\underline{x}\text{-mom}}$ as

$$\nabla\left(\frac{1}{2}\underset{\sim}{u}\cdot\underset{\sim}{u} + \frac{p}{\rho} + gz\right) = -\underset{\sim}{\omega}\times\underset{\sim}{u}$$

Then take a <u>path integral</u>



And note, for any scalar field $\varphi(\underset{\sim}{x})$

$$\int_P \nabla\varphi\cdot d\underset{\sim}{l} = \varphi\Big|_{\underset{\sim}{x}_B} - \varphi\Big|_{\underset{\sim}{x}_A}$$

* assume either (i) $\underset{\sim}{\omega} = 0$ "irrotational flow"

or (ii) path = a streamline

$\Rightarrow$ vector $\underset{\sim}{\omega} \times \underset{\sim}{u}$ is always $\perp$ to path, so $(\underset{\sim}{\omega} \times \underset{\sim}{u}) \cdot d\underset{\sim}{l} = 0$

Then

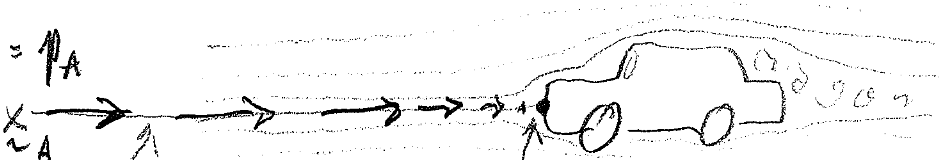
$$\int_P \boxed{\underset{\sim}{x} \text{ mom}} \cdot d\underset{\sim}{l} \Rightarrow \underbrace{\frac{1}{2}\underset{\sim}{u} \cdot \underset{\sim}{u} + \frac{p}{\rho} + gz}_{\text{"Bernoulli Function"}} = \underbrace{\text{constant}}_{\text{on the path, or everywhere if } \underset{\sim}{\omega} = 0}$$

— start here —

Example: Flow around an object ($g = 0$)

$\underset{\sim}{u} = U$, $p = p_A$

$\underset{\sim}{x}_A$



dividing streamline

stagnation point, $\underset{\sim}{x}_B$, $\underset{\sim}{u} = 0$, $p = ?$

$$\left(\frac{1}{2}\underset{\sim}{u} \cdot \underset{\sim}{u} + \frac{p}{\rho} + \cancel{gz}\right)\Big|_{\underset{\sim}{x}_A} = \left(\frac{1}{2}\cancel{\underset{\sim}{u} \cdot \underset{\sim}{u}} + \frac{p}{\rho} + \cancel{gz}\right)\Big|_{\underset{\sim}{x}_B}$$

$$\Rightarrow \frac{1}{2}U^2 + \frac{p_A}{\rho} = \frac{p_B}{\rho} \Rightarrow \boxed{p_B - p_A = \frac{1}{2}\rho U^2}$$ "dynamic pressure"

More NOTES ON BERNOULLI FUNCTION

10/26/2009

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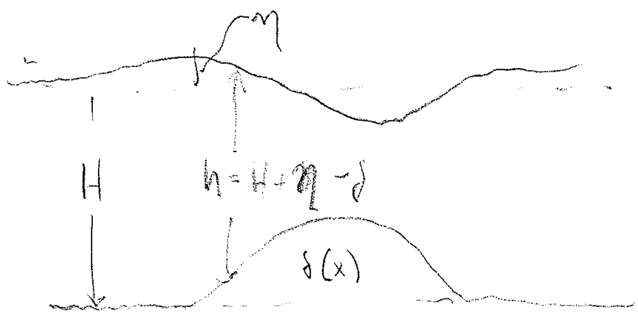

x mom $u u_x = -g\eta_x$

mass $(hu)_x = 0$

$\left.\right\}$ steady, inviscid, ρ = const, hydrostatic $\frac{H}{L} \ll 1$

Bernoulli Function

$\frac{1}{2}\underline{u}\cdot\underline{u} + \frac{p}{\rho} + g\cancel{z} = \text{const}$

$\frac{1}{2}u^2 \quad \frac{p_{atm} + \cancel{\rho} g(\eta - \cancel{z})}{\cancel{\rho}}$

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + g\eta\right) = 0 \quad \xleftrightarrow{\text{same}} \quad \Rightarrow \frac{1}{2}u^2 + g\eta = \text{const}$

so... u faster over bump $\leftrightarrow$ surface depressed

For small perturbations

$u = \mathcal{U} + u' \qquad [u'] \ll \mathcal{U}$

$[\eta] \ll H$

$\Rightarrow \quad \mathcal{U} u'_x = -g\eta_x \quad \Rightarrow \quad \mathcal{U}\,\delta_x \mathcal{U} = -gH\eta_x$

$H u'_x = \delta_x \mathcal{U} \qquad \Rightarrow \quad \eta_x = -\frac{\mathcal{U}^2}{gH}\,\delta_x$