10/20/09 (1)

$\boxed{\text{II.4}}$

SHALLOW WATER WAVES

$$\left.\begin{array}{ll} \boxed{\text{x mom}} & u_t = -g\eta_x \\ \boxed{\text{mass}} & \eta_t + H u_x = 0 \end{array}\right\} \quad \eta_{tt} - gH\eta_{xx} = 0$$

• Note:

$$\left.\begin{array}{l} \dfrac{\partial}{\partial t}\boxed{\text{x mom}} \Rightarrow u_{tt} + g\eta_{xt} = 0 \\ \dfrac{\partial}{\partial x}\boxed{\text{mass}} \Rightarrow \eta_{xt} = -H u_{xx} \end{array}\right\} \Rightarrow u_{tt} - gH u_{xx} = 0$$

We showed any pattern of η moves w/ speed $\pm c = \pm\sqrt{gH}$

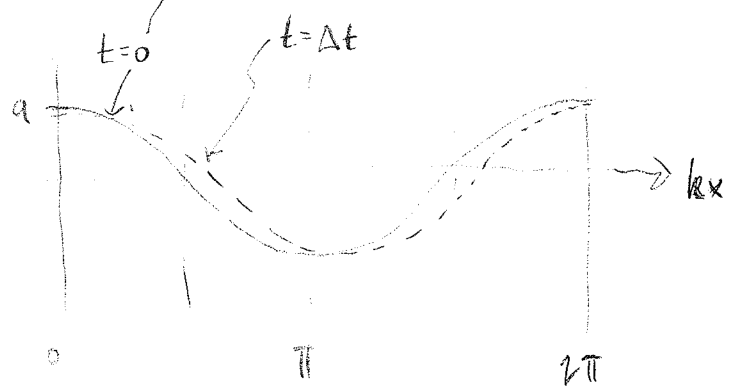
same is true of pattern of u ✓

• Note: including $\boxed{\text{y mom}}$ $\Rightarrow$ $\eta_{tt} - gH(\eta_{xx} + \eta_{yy}) = 0$

• Useful solutions given by the functional form

$$\eta = \underbrace{a\cos(kx - \omega t)} + b\sin(kx - \omega t)$$

• or $\eta = \mathrm{Re}\{\underset{\hat{\eta}\text{ complex}}{\hat{\eta}}\exp i(kx - \omega t)\}$



$\longleftarrow \lambda = \frac{2\pi}{k}$ = wavelength $\longrightarrow$

• pattern moves to right with speed

$$c = \frac{\omega}{k} = \text{"phase speed"}$$

where $\varphi = kx - \omega t$ = "phase"

$$c = \left.\frac{dx}{dt}\right|_{\varphi = \text{const}}$$

$\omega = \dfrac{2\pi}{\text{oscillation period}}$ = frequency $[\text{rad s}^{-1}]$ convention is to assume positive

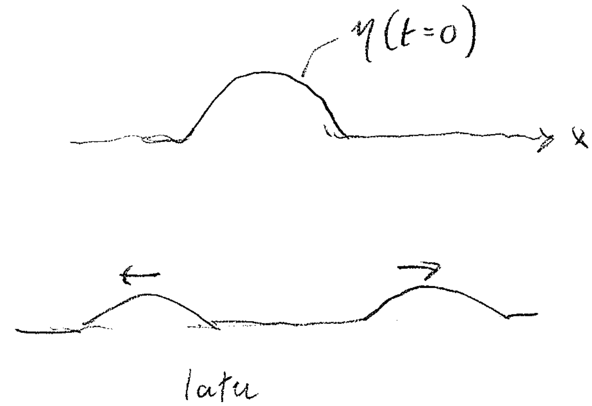
$k = \dfrac{2\pi}{\text{wavelength}}$ = wavenumber $[\text{rad m}^{-1}]$

convention (sometimes) is to allow either sign,
then negative k denotes waves
moving to the left ($c = -\sqrt{gH}$ for the 1-D solution)

Initial / Boundary Conditions (IC + BC)

You need 2 for a 2nd order equation

Example:



IC ① $\boxed{\eta(x, t=0) = \eta_0(x)}$

and $u(x, t=0) = 0$

$\Rightarrow u_x(t=0) = 0$

② $\boxed{\Rightarrow \eta_t(t=0) = 0}$ from $\boxed{\text{mass}}$ $\eta_t + Hu_x = 0$

BC:

e.g. a wall

z, x

$u(x=0, t) = 0$

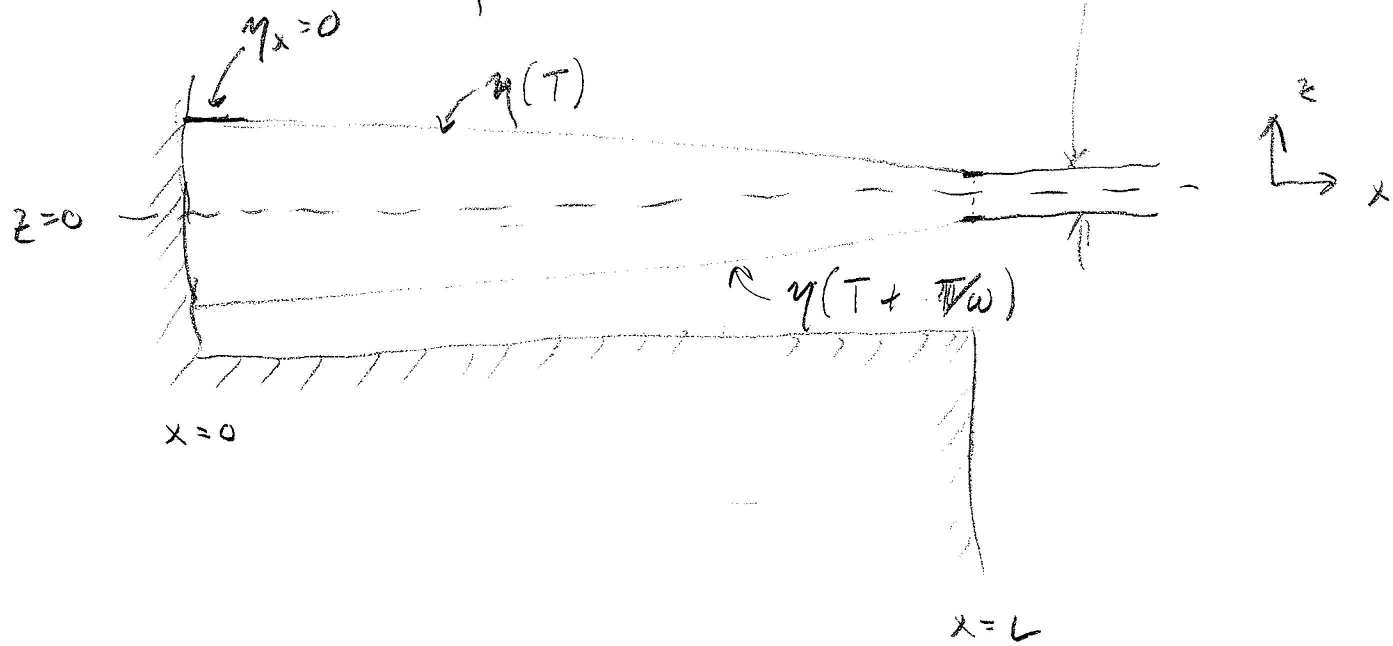
then from $\boxed{x\ \text{mom}}$ $\cancel{u_t} = -g\eta_x$

0

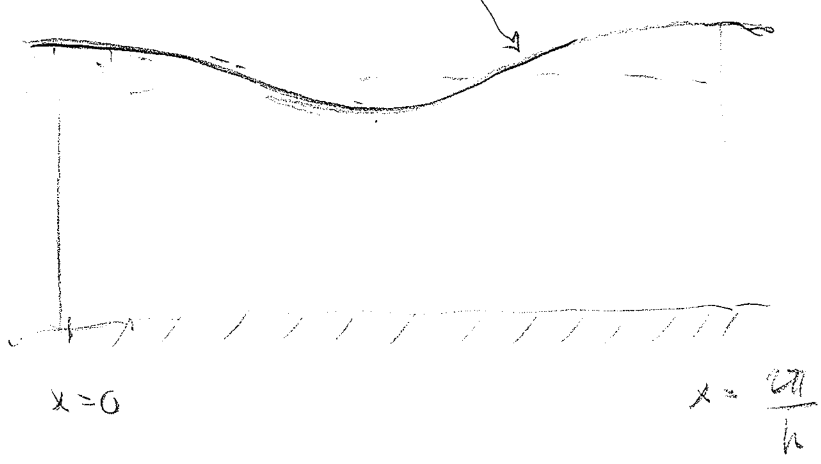
$x=0$

$\Rightarrow \boxed{\eta_x(x=0, t) = 0}$ BC

For the lab experiment the second BC would be $\eta(x=L,t) = \eta_0 \cos \omega t$



Note: Progressive wave: $\eta = \eta_0 \cos(kx-\omega t)$



pattern moves to right

$\longrightarrow$

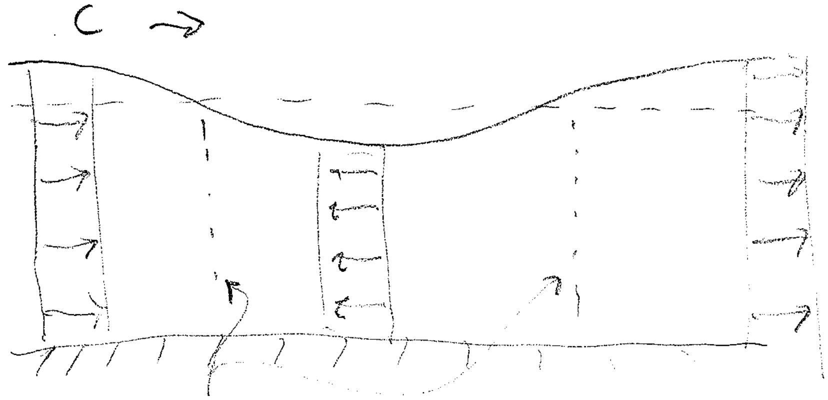
$c = \frac{\omega}{k} = \sqrt{gH}$

$\boxed{x \text{ mom}}$ $u_t = -g\eta_x = -g\eta_0 k(-1)\sin(kx-\omega t)$

$$\Rightarrow \quad u = \int u_t \, dt = \frac{g\eta_0 k(-1)}{(-\omega)} \cos(kx-\omega t) = g\eta_0 \frac{k}{\omega} \cos(kx-\omega t)$$

so $u = \eta_0 g \frac{1}{\sqrt{gH}} \frac{\sqrt{H}}{\sqrt{H}} = \frac{\eta_0}{H} \sqrt{gH} \cos(kx-\omega t)$

so $u = \eta_0 \frac{g}{\sqrt{gH}} \frac{\sqrt{H}}{\sqrt{H}} \cos(kx-\omega t) = \boxed{\frac{\eta_0}{H}\sqrt{gH}\,\cos(kx-\omega t) = u}$



for $\frac{\eta_0}{H} \ll 1$

$\Rightarrow |u| \ll c$

$u=0$

where $|\eta_x| = \max$

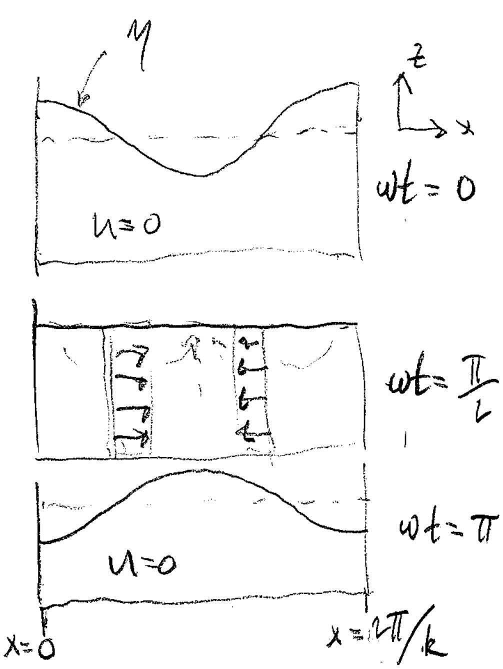
Note: "standing wave" = sum of two equal waves going opposite direction

$$\eta = \overrightarrow{a\cos(kx-\omega t)} + \overleftarrow{a\cos(kx+\omega t)}$$

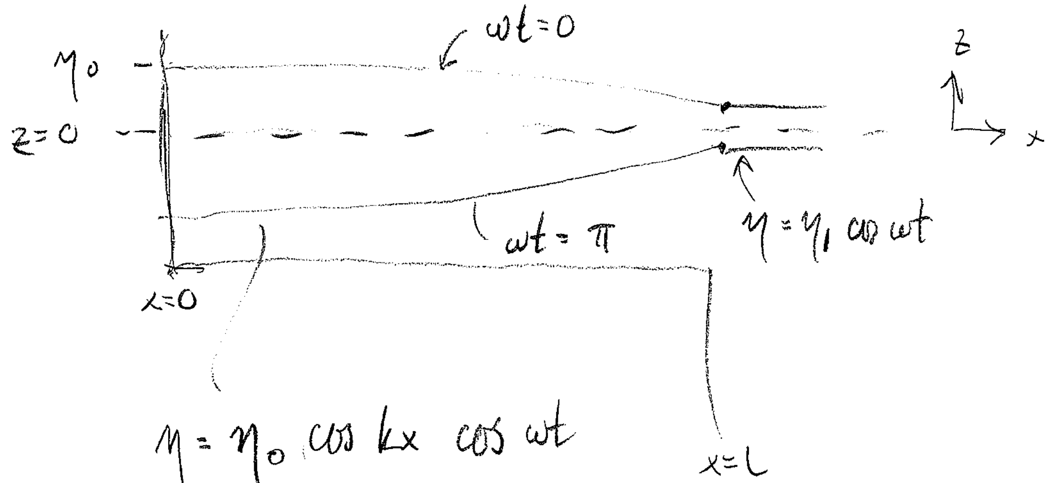
$$= a\cos kx\cos\omega t + \cancel{a\sin kx\sin\omega t}$$

$$+ a\cos kx\cos\omega t - \cancel{a\sin kx\sin\omega t}$$

$$= 2a\cos kx\cos\omega t$$



e.g. for the lab experiment



$$\eta = \eta_0 \cos kx \cos \omega t$$

satisfies BC 1 : $\eta_x (x=0) = 0$

to satisfy BC 2 $\Rightarrow$

$$\eta_0 \cos kL \cancel{\cos \omega t} = \eta_1 \cancel{\cos \omega t}$$

'$\frac{1}{4}$ wave oscillator'

$$\Rightarrow \eta_0 = \frac{\eta_1}{\cos(kL)}$$