SHALLOW WATER WAVES $\boxed{\text{II}.3}$

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Start from our Shallow Water Equations $\left(\frac{\partial}{\partial y} = 0\right)$ "1D"

$\boxed{x \text{ mom}}$ $u_t + u u_x = -g\eta_x$ *

$\boxed{\text{mass}}$ $\eta_t + (hu)_x = 0$ $\quad h = H + \eta$

} nonlinear *

Consider small perturbations away from a state of rest

$\Rightarrow$ allows us to linearize

Consider $\boxed{\text{mass}}$ + scale $[\eta] = \varepsilon$, assume $\varepsilon \ll H$

$$\left[\ \eta_t + H u_x + \eta u_x + \cancelto{0}{H_x} u + \eta_x u = 0\ \right]$$

$$\frac{\varepsilon}{T} \qquad \frac{H\mathcal{U}}{L} \qquad \frac{\varepsilon\,\mathcal{U}}{L} \qquad \frac{\varepsilon\,\mathcal{U}}{L}$$

$$(1) \qquad (2) \qquad (3) \qquad (4)$$

(2) >> (3) a (4) so drop (3) + (4) ✓

leaves $\boxed{\text{mass}}$ $\boxed{\eta_t + H u_x = 0}$ linear!

Wait! Previously we assumed "advective scaling" $\Rightarrow \frac{1}{T} \sim \frac{\mathcal{U}}{L}$

which would mean (1) was also negligible!

But ... many important <u>wave</u> flows (Tides, Kelvin Waves, etc.) have constant T (wave period) ②

L (wavelength)

even as $\mathcal{U} \to 0$

$\Rightarrow$ for waves $\frac{1}{T}$ may be $\gg \frac{\mathcal{U}}{L}$!

so we retain term ① (means we require $\frac{1}{T} \gg \frac{\mathcal{U}}{L}$)

Next, consider $\boxed{x \text{ mom}}$

$$\left[u_t + u u_x = -g \eta_x \right]$$

$$\Rightarrow \quad \underset{①}{\frac{\mathcal{U}}{T}} \quad \underset{②}{\frac{\mathcal{U}^2}{L}} \quad \underset{③}{g\frac{\varepsilon}{L}}$$

① $\gg$ ② since $\frac{1}{T} \gg \frac{\mathcal{U}}{L}$ $\Rightarrow$ drop ②

leaves

$\boxed{x \text{ mom}}$ $\quad u_t = -g\eta_x$

$\boxed{\text{mass}}$ $\quad \eta_t + H u_x = 0$

2 equations in u, η

Take $\frac{\partial}{\partial t}\boxed{\text{mass}} \Rightarrow \eta_{tt} + H u_{xt} = 0$

$\frac{\partial}{\partial x}\boxed{\text{x mom}} \Rightarrow u_{xt} = -g\eta_{xx}$

$$\Rightarrow \boxed{\eta_{tt} - gH\eta_{xx} = 0}$$

a <u>wave equation!</u>

this admits a general solution of the form

$\eta = f(\xi)$ where $\xi = x \pm ct$

so $\frac{\partial \eta}{\partial t} = \frac{df}{d\xi}\frac{\partial \xi}{\partial t} = \pm f'c \Rightarrow \eta_{tt} = f''c^2$

$\frac{\partial \eta}{\partial x} = \frac{df}{d\xi}\frac{\partial \xi}{\partial x} = f' \Rightarrow \eta_{xx} = f''$

so $\cancel{f''}c^2 - gh\cancel{f''} = 0$

$\Rightarrow c = \sqrt{gH}$ (take positive root)