$\boxed{II.1}$ "Hydrostatic" Holton 2.4.3 "An Introduction to Dynamic Meteorology" 1979

Boussinesq KC 4.18

$\boxed{\text{SCALING}}$

Q: when is $\frac{\partial p}{\partial z} = -\rho g$ a good idea?

Equations

$\boxed{x, y \text{ mom}}$ $\rho \frac{D \underset{\sim}{u}_H}{Dt} = -\nabla_H p$

$\underset{\sim}{u}_H = (u, v)$

$\nabla_H = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$\boxed{z \text{ mom}}$ $\rho \cdot \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \rho g$

$\boxed{\text{mass}}$ $\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla_H \cdot \underset{\sim}{u}_H + \frac{\partial w}{\partial z} = 0$

let $[\star]$ denote the "scale of $\star$" and let

$$\rho = \rho_0 + \rho'(\underset{\sim}{x}, t)$$

strictly speaking this would also have to include the free surface in order for

$$p = \bar{p}(z) + p'(\underset{\sim}{x}, t)$$

assume that we <u>observe</u>

velocity scale $[\underset{\sim}{u}_H] = \mathcal{U}$

horiz. length scale $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = \frac{1}{L}$

vertical length scale $\left[\frac{\partial}{\partial z}\right] = \frac{1}{H}$

* For most important atm/ocn flows we assume $H \ll L$

Define $[w] = W$ unknown so far

Observe $[\rho'] = \rho_1$

* assume $\rho_1 \ll \rho_0$ (when is this <u>not</u> a good idea?)

"Boussinesq approximation":

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* assume $\left[\frac{\partial}{\partial t}\right] = \frac{\mathcal{U}}{L}$ "advective time scale"

scaling $\boxed{\text{mass}}$

$$\left[\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \underset{\sim}{u} = 0 \right]$$

$$\Rightarrow \left[\underbrace{\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}(u\rho_x + v\rho_y)}_{\frac{\rho_1}{\rho_0}\frac{\mathcal{U}}{L}} + \underbrace{\frac{1}{\rho}w\rho_z}_{\frac{\rho_1}{\rho_0}\frac{W}{H}} + \underbrace{(u_x + v_y)}_{\frac{\mathcal{U}}{L}} + \underbrace{w_z}_{\frac{W}{H}} = 0 \right]$$

(1) (2) (3) (4)

(1) $\ll$ (3) and (2) $\ll$ (4) because $\frac{\rho_1}{\rho_0} \ll 1$

$\therefore$ $\boxed{\text{mass}}$ is approximated by $\boxed{\nabla \cdot \underset{\sim}{u} = 0}$

and $\frac{\mathcal{U}}{L} = \frac{W}{H} \Rightarrow W = \mathcal{U}\frac{H}{L}$ $\left(\text{and so } \left[\frac{D}{Dt}\right] = \frac{\mathcal{U}}{L}\right)$

Then, scaling $\boxed{x, y \text{ mom}}$

$$\left[\rho \frac{D\underset{\sim}{u}_H}{Dt} = -\nabla_H p \right]$$

$$\Rightarrow \quad \rho_0 \frac{\mathcal{U}^2}{L} = \frac{[p']}{L} \quad \Rightarrow \quad \boxed{[p'] = \rho_0 \mathcal{U}^2} \quad (++)$$

like "dynamic pressure"

— Start here

Finally, scale $\boxed{z\text{-mom}}$

NOTE: we can define $\bar{p}$ as being in hydrostatic balance with ρ_0: so $\bar{p}_z = -\rho_0 g$ $(*)$

$$\left[\rho \frac{Dw}{Dt} = \underbrace{-\cancel{\bar{p}_z} - p'_z - \cancel{\rho_0 g}}_{\text{cancel, by } (*)} - \rho' g \right]$$

$$\text{scaling} \quad \rho_0 \frac{W\mathcal{U}}{L} = \frac{[p']}{H} - [\rho'] g$$

$$\text{or} \quad \rho_0 \frac{W\mathcal{U}}{L} \qquad \frac{\rho_0 \mathcal{U}^2}{H} \qquad \rho_1 g$$

1 2 3

using (++)

and $\frac{①}{②} = \frac{W U}{L} \frac{H}{U^2} = \left(\frac{H \cancel{U}}{L}\right) \frac{\cancel{U} H}{L \cancel{U^2}} = \frac{H^2}{L^2}$

so, if $H \ll L$ ① is truly negligible compared to ②

leaving only ③ to balance ②

or $\frac{\partial p'}{\partial z} = -\rho' g$ $\left(\Rightarrow \frac{\partial p}{\partial z} = -\rho g \text{ since } \frac{\partial \bar{p}}{\partial z} = -\rho_0 g \text{ as well} \right)$

Summarizing, we have $\left(\text{note: } \frac{1}{\rho}\frac{Du}{Dt} \cong \frac{1}{\rho_0}\frac{Du}{Dt}\right)$

		$[\rho'] \ll \rho_0$	$H \ll L$
$\boxed{x \text{ mom}}$	$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$	✓	
$\boxed{y \text{ mom}}$	$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$	✓	
$\boxed{z \text{ mom}}$	$0 = -\frac{\partial p}{\partial z} - \rho g$	(✓)	✓
$\boxed{\text{mass}}$	$\nabla \cdot \underline{u} = 0$	✓	
		Boussinesq	Large aspect ratio