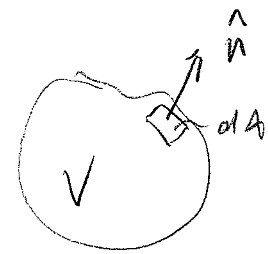
Volume integrals

III.4 10/29/2009

- useful for understanding how whole fluid systems work
  e.g. momentum or energy budget

- make use of Gauss Divergence Theorems

$$\int_V \nabla\cdot \underset{\sim}{c}\, dV = \int_A \underset{\sim}{c}\cdot \hat{n}\, dA \qquad (*)$$



$$\int_V \nabla\varphi\, dV = \int_A \varphi\, \hat{n}\, dA \qquad (**)$$

- e.g. mass conservation

$\boxed{\text{Mass}}$ $\quad \frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot \underset{\sim}{u} = 0$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \underbrace{\underset{\sim}{u}\cdot\nabla\rho + \rho\,\nabla\cdot\underset{\sim}{u}} = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot(\underset{\sim}{u}\rho) = 0$$

then take vol $\int$al
over a volume V fixed in space

$$\int_V \frac{\partial\rho}{\partial t}\, dV = \frac{d}{dt}\int_V \rho\, dV = -\int_V \nabla\cdot(\underset{\sim}{u}\rho)\, dV \underset{\substack{\uparrow \\ \text{use } (*)}}{=} -\int_A \rho\, \underset{\sim}{u}\cdot\hat{n}\, dA$$

thus

$$\frac{d}{dt}\int_V \rho \, dV = -\int_A \rho \, \underset{\sim}{u} \cdot \hat{n} \, dA$$

rate of change of total mass in V = − net advective flux of mass out of the volume

$$\text{vocabulary note "advective mass flux"} \sim \frac{kg}{s\,m^2} = \underbrace{\frac{kg}{m^3}}_{\rho} \underbrace{\overbrace{\frac{m^3}{s}}^{\text{volume transport}} \overbrace{\frac{1}{m^2}}^{\text{per unit area}}}_{\underset{\sim}{u}\cdot\hat{n}}$$

How about momentum?

SIMPLE CASE

consider just x-mom, ρ = const.

$\boxed{\text{x mom}}$ $\qquad \underbrace{\frac{Du}{Dt}}_{\frac{\partial u}{\partial t} + \underset{\sim}{u} \cdot \nabla u} = -\frac{1}{\rho} p_x + \underbrace{\nu \nabla^2 u}_{\nu \nabla \cdot \nabla u}$

$u = \frac{\text{x momentum}}{\text{unit mass}}$, we want $\rho u = \frac{\text{x momentum}}{\text{unit vol.}}$ so

so multiply by ρ, and add $\rho u (\nabla \cdot \underset{\sim}{u})$ $= 0$

$$\frac{\partial \rho u}{\partial t} + \underbrace{\underset{\sim}{u} \cdot \nabla(\rho u) + \rho u (\nabla \cdot \underset{\sim}{u})}_{\nabla \cdot (\underset{\sim}{u}\, \rho u)} = \underbrace{- p_x}_{-\hat{i} \cdot \nabla p} + \mu \nabla \cdot \nabla u$$

take volume integral

$$\int_V (\rho u)_t \, dV = -\int_V \nabla \cdot (\underset{\sim}{u}\, \rho u)\, dV - \hat{i} \cdot \int_V \nabla p \, dV + \mu \int_V \nabla \cdot \nabla u \, dV$$

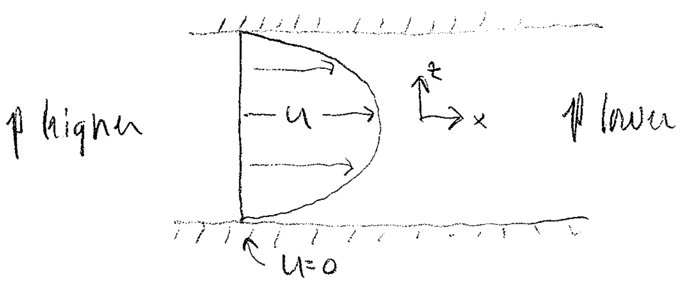
use (**)

use (*)

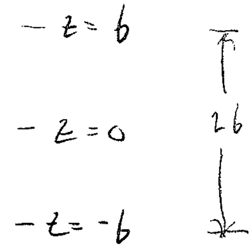
$$\Rightarrow \frac{d}{dt}\int_V \rho u \, dV = -\int_A (\rho u)\, \underset{\sim}{u}\cdot\hat{n}\, dA - \int_A p\, \hat{i}\cdot\hat{n}\, dA + \mu\int_A \nabla u\cdot\hat{n}\, dA$$

net rate of change of x-mom in V = advective flux of x-mom through boundary + net force due to pressure (normal stress) + net force due to friction (tangential stress)

Example Plane Poiseuille Flow ⟨KC 9.4⟩



$\frac{dp}{dx} < 0$



$$0 = -p_x + \mu u_{zz} \quad \Rightarrow \quad u_{zz} = \mu p_x \quad \text{(a constant)}$$

$$\Rightarrow \quad u = \frac{1}{\mu} p_x \cdot \frac{1}{2} z^2 + Az + B$$

BC's $\quad u(z = \pm b) = 0$

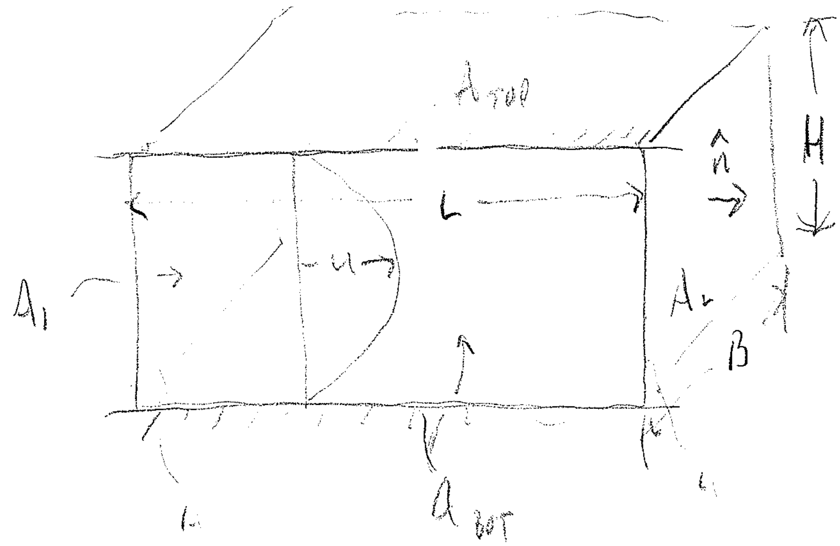
$$\left.\begin{aligned} \Rightarrow (z=b) \quad & 0 = \frac{1}{\mu} p_x \frac{1}{2} b^2 + Ab + B \\ (z=-b) \quad & 0 = \frac{1}{\mu} p_x \frac{1}{2} b^2 - Ab + B \end{aligned}\right\} \text{subtract} \Rightarrow \quad A = 0$$

Add $\Rightarrow \; 2B = -\frac{1}{\mu} p_x b^2$

$$\therefore \quad u = \frac{1}{\mu} p_x \left(\frac{1}{2} z^2 - \frac{1}{2} b^2 \right) \Rightarrow -\left(\frac{1}{2} \frac{p_x b^2}{\mu} \right) \left[1 - \left(\frac{z}{b} \right)^2 \right]$$

$$\frac{\partial u}{\partial z} = \left(-\frac{1}{\cancel{2}} \frac{p_x b^2}{\mu} \right) \left[-\frac{\cancel{2} z}{\cancel{b^2}} \right] = +\frac{1}{\mu} p_x z$$

$$u_{zz} = \frac{1}{\mu} p_x \quad \Rightarrow \quad 0 = -p_x + \mu u_{xx} \checkmark$$



$V = LHB$

$$\frac{d}{dt}\int \rho u \, dV = 0$$

$$-\int_A \rho u \, \underline{u}\cdot\hat{n} \, dA = -\int_{A_1} \rho u \underbrace{(-u)}_{\underline{u}\cdot\hat{n} \text{ on } A_1} dA - \int_{A_2} \rho u \underbrace{(u)}_{\underline{u}\cdot\hat{n} \text{ on } A_2} dA = 0$$

$$-\int_A p \, \hat{i}\cdot\hat{n} \, dA = -p_1 A_1(-1) - p_2 A_2(1) = -\overbrace{HB\left(L\frac{\partial p}{\partial x}\right)}^{-V p_x}$$

$$\mu \int_A \nabla u \cdot \hat{n} \, dA = \mu \int_{A_{BOT}} \left(-\frac{\partial u}{\partial z}\right)\bigg|_{z=-b} dA + \mu \int_{A_{TOP}} \left(\frac{\partial u}{\partial z}\right)\bigg|_{z=b} dA$$

$$= \mu BL\left(\frac{-1}{\mu} p_x\right)(-b) + \mu BL\left(\frac{1}{\mu} p_x\right) b$$

$$= BL \underbrace{(2b)}_{H} p_x = V p_x$$

III

So the total force balance is

net rate of change of x-momentum in V	=	net pressure force on open ends	+	net viscous stress on top + bottom walls	[N]
$\emptyset$	=	$(+)$		$(-)$	