FLUID ENERGY $\left[\frac{kg\,m^2}{s^2} = \text{Joule} = J\right]$ III.3

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Conceptually separated into two categories

1. Internal Energy = KE of random molecular motions

c_p = specific heat at const. pressure: dry air $c_p = 1012\ J/kg\,K$; water $c_p = 4182\ J/kg\,K$

$K = (°C + 273.15)$

2. Mechanical Energy: Kinetic & Potential Energies of macroscopic fluid properties $(\underset{\sim}{u}, e)$

$$\text{Energy} = \text{Work Done} = \int \underbrace{\underset{\sim}{F} \cdot \underset{\sim}{u}}_{\text{rate of doing work } [J\,s^{-1} = \text{Watt} = W]}\, dt$$

where $\underset{\sim}{F}$ is the net force exerted on a mass m moving at velocity $\underset{\sim}{u}$ (not a fluid here)

$$m\underset{\sim}{a} = \underset{\sim}{F} - \hat{k} g m \qquad \text{equation of motion for } m$$

$$\Rightarrow \underset{\sim}{F} = m\frac{d\underset{\sim}{u}}{dt} + \hat{k} g m \Rightarrow \underset{\sim}{F} \cdot \underset{\sim}{u} = m \underbrace{\underset{\sim}{u} \cdot \frac{d\underset{\sim}{u}}{dt}}_{\frac{d}{dt}\left(\frac{1}{2}\underset{\sim}{u}\cdot\underset{\sim}{u}\right)} + m g \underbrace{w}_{\frac{dz}{dt}}$$

So $\text{Energy} = \int \underset{\sim}{F} \cdot \underset{\sim}{u} \, dt = \underbrace{\frac{1}{2} m \underset{\sim}{u} \cdot \underset{\sim}{u}}_{\text{Kinetic Energy: KE}} + \underbrace{mgz}_{\text{Potential Energy: PE}} + \text{Const}$ (related to choice of $z=0$)

For fluids we use

$$KE_V \equiv \frac{KE}{\text{unit volume}} = \frac{1}{2} \rho \underset{\sim}{u} \cdot \underset{\sim}{u}$$

$$KE_M \equiv \frac{KE}{\text{unit mass}} = \frac{1}{2} \underset{\sim}{u} \cdot \underset{\sim}{u}$$

$$PE_V \equiv \frac{PE}{\text{unit volume}} = \rho g z$$

$$PE_M \equiv \frac{PE}{\text{unit mass}} = gz$$

Develop an equation for KE :

$\mathbf{u} \cdot \boxed{\mathbf{x} \text{ mom}}$ $\quad \mathbf{u} \cdot \left[\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p - \hat{k} g + \nu \nabla^2 \mathbf{u} \right]$

$$\qquad (1) \qquad\qquad (2) \qquad\qquad (3) \qquad\qquad (4)$$

term by term:

(1) $\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u}\right) + \mathbf{u} \cdot (\boldsymbol{\omega} \times \mathbf{u})$

0 because $\boldsymbol{\omega} \times \mathbf{u}$ is $\perp$ to $\mathbf{u}$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u}\right) + \mathbf{u} \cdot \nabla\left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u}\right)$$

$$= \frac{D}{Dt}\left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u}\right)$$

(2) $\mathbf{u} \cdot \left(-\frac{1}{\rho} \nabla p\right) = -\frac{1}{\rho} \nabla \cdot (\mathbf{u} p) + \frac{p}{\rho}(\nabla \cdot \mathbf{u})$

(3) $\mathbf{u} \cdot (-\hat{k} g) = -g w$

(4) $\mathbf{u} \cdot (\nu \nabla^2 \mathbf{u}) = \underbrace{\nu u_i \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j}}_{\text{use indicial notation KC 2}} = \underbrace{\nu \frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_i}{\partial x_j}\right)}_{(a)} \underbrace{- \nu \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}\right)}_{(b)}$

(a) : $= \nu \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}\left(\frac{1}{2} u_i u_i\right)\right] = \nu \nabla^2 \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u}\right)$

(b) $= -\nu\left[(u_x)^2 + (u_y)^2 + (u_z)^2 + (v_x)^2 + \ldots (w_z)^2\right] \equiv -\varepsilon$

So putting it back together

$\boxed{KE_M}$ $[W\ kg^{-1}]$

$$\boxed{\frac{D}{Dt}(KE_M) = -\frac{1}{\rho}\nabla\cdot(\underset{\sim}{u}\,p) + \frac{p}{\rho}(\nabla\cdot\underset{\sim}{u}) - gw + \underbrace{\nu\nabla^2(KE_M)}_{\text{viscous redistribution}} - \underbrace{\varepsilon}_{\text{rate of loss of } KE_M}}$$

viscous redistribution of KE_M, or addition of KE_M by viscous stress at a moving boundary

rate of loss of KE_M (goes to internal energy)

"Dissipation"

Note ε is positive definite

Often we use volume integrals for energy budgets – then it is convenient to start with an equation for $KE_V = \frac{1}{2}\rho\,\underset{\sim}{u}\cdot\underset{\sim}{u}$

Note, for any property α

$$\rho\frac{D\alpha}{Dt} = \rho\alpha_t + \rho\,\underset{\sim}{u}\cdot\nabla\alpha + \underbrace{\alpha\rho_t + \alpha\nabla\cdot(\rho\underset{\sim}{u})}_{=0 \text{ from } \alpha\ \boxed{\text{MASS}}}$$

$$= (\rho\alpha)_t + \nabla\cdot(\underset{\sim}{u}\,\rho\alpha)$$

Applying this where $\alpha = KE_m$, $\rho\alpha = KE_v$

ignore ρ gradients at viscous scale

$$\rho\,\boxed{KE_m} \quad \rho \frac{D}{Dt}(KE_m) = -\nabla\cdot(\underline{u}\,p) + p(\nabla\cdot\underline{u}) - \rho g w + \nu \nabla^2(\rho\, KE_m) - \rho\,\varepsilon$$

becomes

$$\boxed{KE_v} \quad \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho\,\underline{u}\cdot\underline{u}\right) = -\nabla\cdot\left[\underline{u}\left(\tfrac{1}{2}\rho\,\underline{u}\cdot\underline{u}\right)\right] - \nabla\cdot(\underline{u}\,p) + p(\nabla\cdot\underline{u})$$

$$- \rho g w + \nu\,\nabla\cdot\nabla\left(\tfrac{1}{2}\rho\,\underline{u}\cdot\underline{u}\right) - \rho\,\varepsilon$$