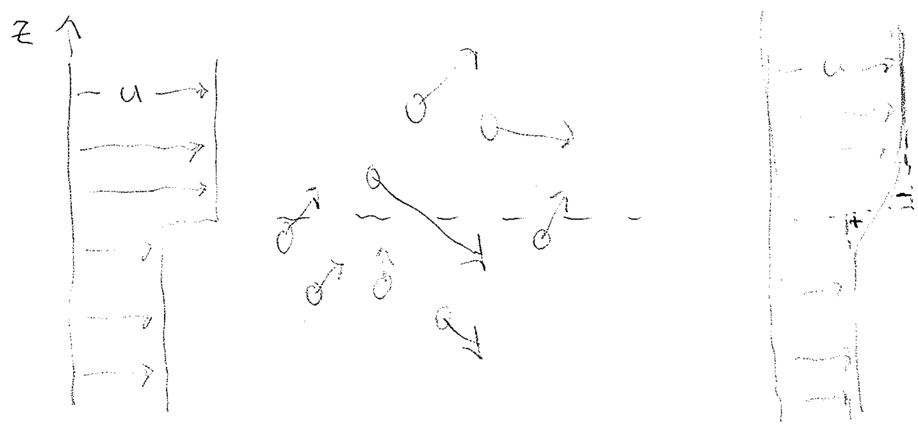
III.2 VISCOSITY continued...

You can also think of this as arising from molecular diffusion of momentum:



$t = 0$ $\qquad$ $t =$ later

NOTE: "flux": rate of advection/diffusion/transmission of a property across a surface, per unit area

$$z\text{-direction flux of } x\text{-momentum} = -\mu \frac{\partial u}{\partial z} \quad \Big[\underbrace{\frac{kg\,m}{s}}_{\text{property is momentum}} \; \underbrace{\frac{1}{s}}_{\text{rate}} \; \underbrace{\frac{1}{m^2}}_{\text{per unit area}}\Big] = \Big[\frac{kg}{m\,s^2}\Big] = \Big[\frac{N}{m^2}\Big]$$

more generally define

$$\underset{\sim}{\mathbb{F}}^{x} = \text{direction flux of } x\text{-momentum} = -\mu \nabla u$$

then the viscous force on a parcel is

$$\text{Visc. Force}^{x} = \underbrace{\int_A -\underset{\sim}{\mathbb{F}}^{x} \cdot \hat{n}\, dA}_{\text{net inward flux of momentum (or net viscous stress on A)}} \overset{\text{using Gauss Divergence Thm.}}{=} \int_V -\nabla \cdot \underset{\sim}{\mathbb{F}}^{x}\, dV$$

and in the limit $V \to 0$

$$\frac{\text{Visc. Force}^{x}}{\text{unit volume}} = -\nabla \cdot \underset{\sim}{\mathbb{F}}^{x} = \mu \nabla \cdot \nabla u = \mu \nabla^2 u \quad \text{again}$$

Summarizing: the equation of motion may be written as

$$\boxed{x \text{ mom}} \quad \frac{Du}{Dt} = -\frac{1}{\rho} p_x + \nu \nabla^2 u$$

$$\boxed{y \text{ mom}} \quad \frac{Dv}{Dt} = -\frac{1}{\rho} p_y + \nu \nabla^2 v$$

$$\boxed{z \text{ mom}} \quad \frac{Dw}{Dt} = -\frac{1}{\rho} p_z - g + \nu \nabla^2 w$$

conservation of $\frac{\text{momentum}}{\text{unit mass}}$

$$\boxed{\text{mass}} \quad \frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \underset{\sim}{u} = 0$$

where $\nu \equiv \frac{\mu}{\rho}$ = "kinematic viscosity" $[m^2 s^{-1}]$

at 20 °C, $p = 1$ atm.

	ν $[m^2 s^{-1}]$	ρ $[kg\ m^{-3}]$
air	1.5×10^{-5}	~ 1.2
water	1.0×10^{-6}	~ 1000

Note: a complete derivation includes a viscous term proportional to $\nabla \cdot \underset{\sim}{u}$ but few flows have viscosity & compressibility important at the same time

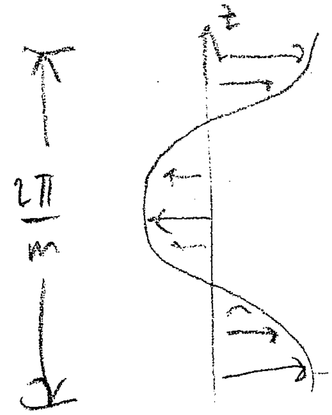
Exact: KC 4.6 stress tensor

KC 4.7 mom. conservation

KC 4.10 viscosity w/ $\nabla \cdot \underset{\sim}{u} \neq 0$ for a true Newtonian fluid

$\boxed{\text{Example}}$: viscosity gets rid of variations in velocity

consider a flow like



$\frac{\partial}{\partial x}, \frac{\partial}{\partial y} = 0$

$v, w = 0$

$\boxed{x \text{ mom}}$

$$\frac{\partial u}{\partial t} + u \cancel{u_x} + v \cancel{u_y} + \cancel{w} u_z = -\frac{1}{\rho} \cancel{p_x} + \nu\left(\cancel{u_{xx}} + \cancel{u_{yy}} + u_{zz}\right)$$

$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

so $\quad u_t = \nu u_{zz}$ (like a "heat equation")

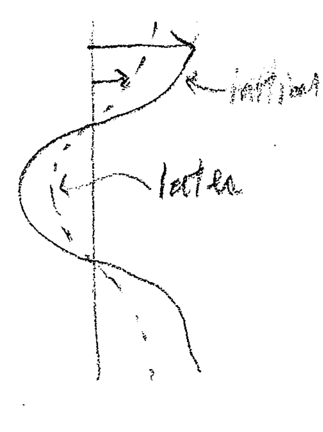
if $\quad u(t=0) = u_0 \cos(mz)$

guess $\quad u = \mathcal{U}(t) \cos(mz)$ [trying separation of variables]

$\Rightarrow \quad \frac{\partial \mathcal{U}}{\partial t} + \nu m^2 \mathcal{U} = 0$

$\Rightarrow \quad \mathcal{U} = u_0 e^{-\nu m^2 t}$

full solution $\quad u = u_0 e^{-\nu m^2 t} \cos(mz)$

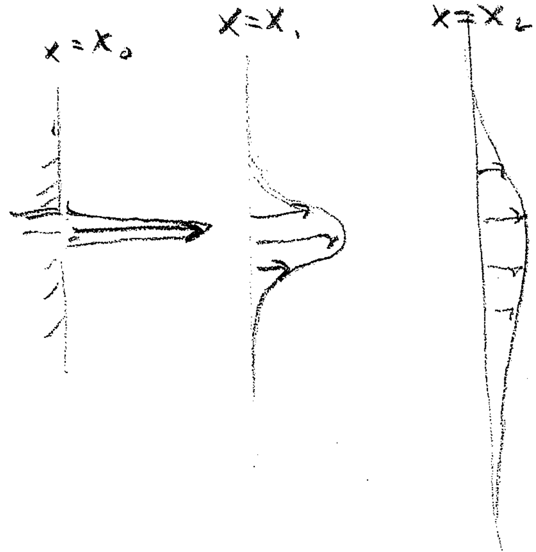


– viscosity weakens the layered flow

(4)

Example 2

Jet



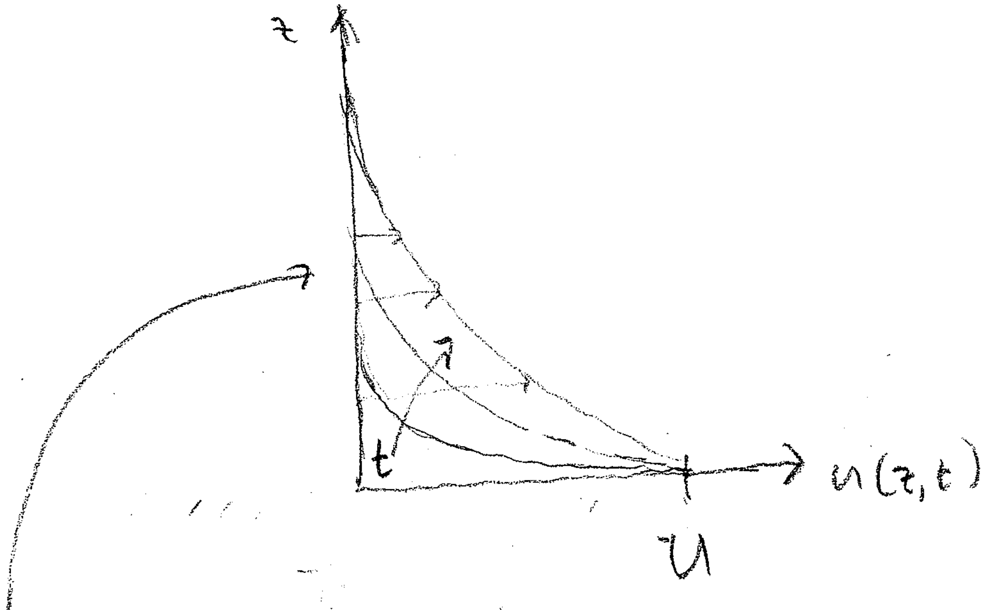
Viscosity makes jet grow broader and weaker

Example 3

Boundary Layer: start moving a boundary

$u = 0$ initially

$U \Rightarrow$



region of viscous <u>acceleration</u> of fluid

grows thicker over time

scaling $\quad u_t = \nu u_{zz} \quad \Rightarrow \quad L \propto \sqrt{\nu t}$

$\frac{U}{T}$

For water at $20°C$ $\nu = 10^{-6}\ m^2\ s^{-1}$

so in 10^2 sec

$$L = \sqrt{10^{-6}\ \frac{m^2}{s}\ 10^2\ s} = \sqrt{10^{-4}\ m^2} = 1\ cm$$

(note: doesn't depend on the actual velocity!)