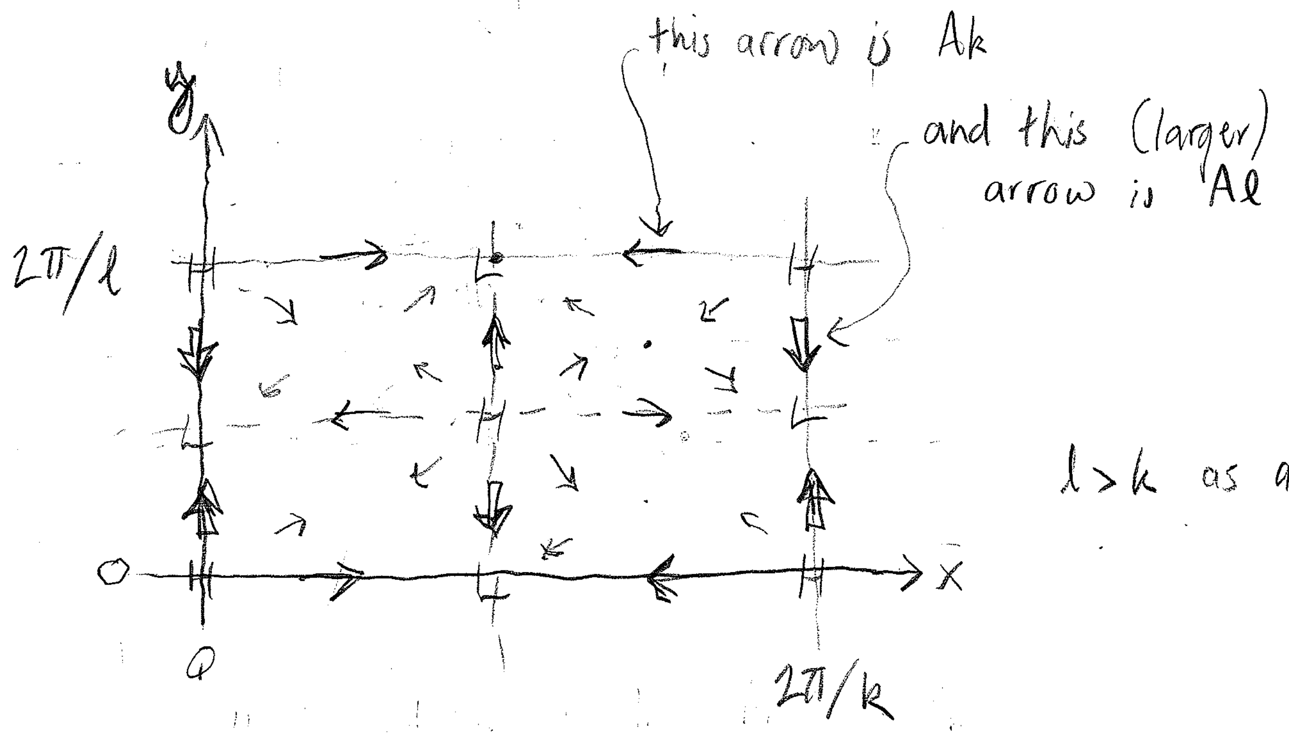
FLUIDS 2009

Problem Set #2 Solutions

1.i. Since $p = A\cos(kx)\cos(ly)$

$$\Rightarrow -\nabla p = \left(-\frac{\partial p}{\partial x}, -\frac{\partial p}{\partial y}\right) = \left(Ak\sin(kx)\cos(ly),\ Al\cos(kx)\sin(ly)\right)$$



$l > k$ as drawn

H & L denote the initial pressure field

1.ii. The divergence is

$$-\nabla\cdot\nabla p = \left(-\frac{\partial}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial}{\partial y}\frac{\partial p}{\partial y}\right) = -\left(p_{xx} + p_{yy}\right) = -\nabla^2 p$$

$$= A(k^2 + l^2)\cos kx \cos ly$$

1.iii. Clearly this would not be incompressible if $\underset{\sim}{u} \propto -\nabla p$

2.i. $\frac{DT}{Dt} = 0$

2.ii. Zero

2.iii. Note that $\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0 \quad (*)$

Does $\frac{\partial T}{\partial x}$ change with time? No. you can show this by taking $\frac{\partial (*)}{\partial x}$ to form

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial x}\right) + \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0} \frac{\partial T}{\partial x} + \underbrace{u \cancel{\frac{\partial^2 T}{\partial x^2}}}_{0 \text{ at } t=0} = 0$$

$\therefore \frac{\partial T}{\partial x}$ is constant for all time, and $\frac{\partial T}{\partial x} = \frac{T_0}{L}$

so $(*)$ can be written as

$$\frac{\partial T}{\partial t} + \frac{u}{H}\frac{T_0}{L} z = 0 \Rightarrow T_t = -\frac{u T_0}{HL} z$$

this has solution

$$T = -\frac{U T_0}{HL} z t + \overbrace{f(x,z)}^{T(t=0)}$$

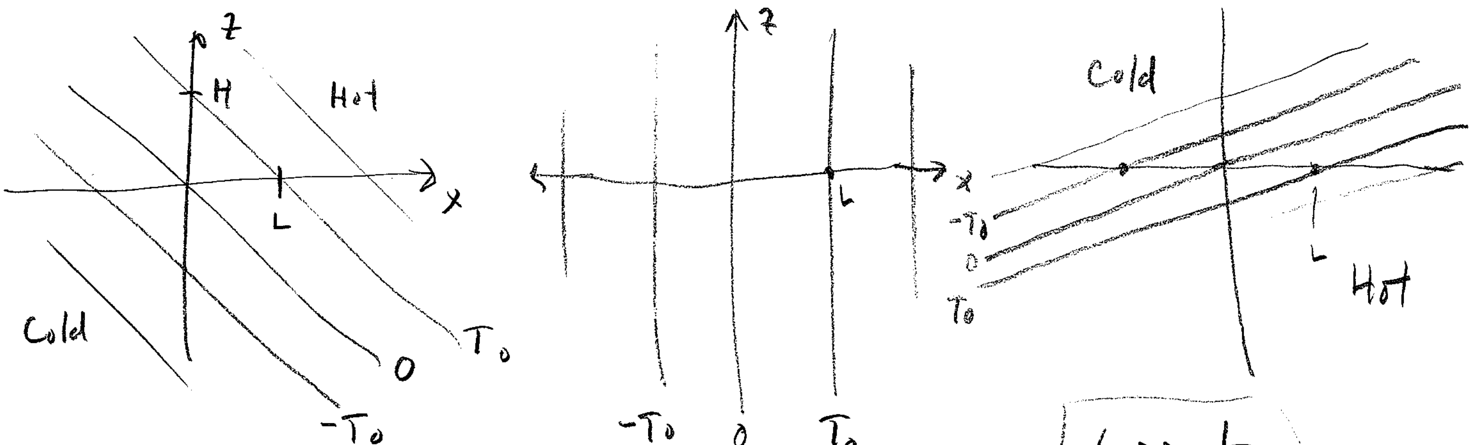
constant of integration

so the full solution is

$$\boxed{T = \frac{T_0}{L} x + \frac{T_0}{H} z \left(1 - \frac{U}{L} t\right)}$$

at $z = \frac{H}{2}$:

$$\frac{\partial T}{\partial t} = -\frac{T_0 U}{2L}$$

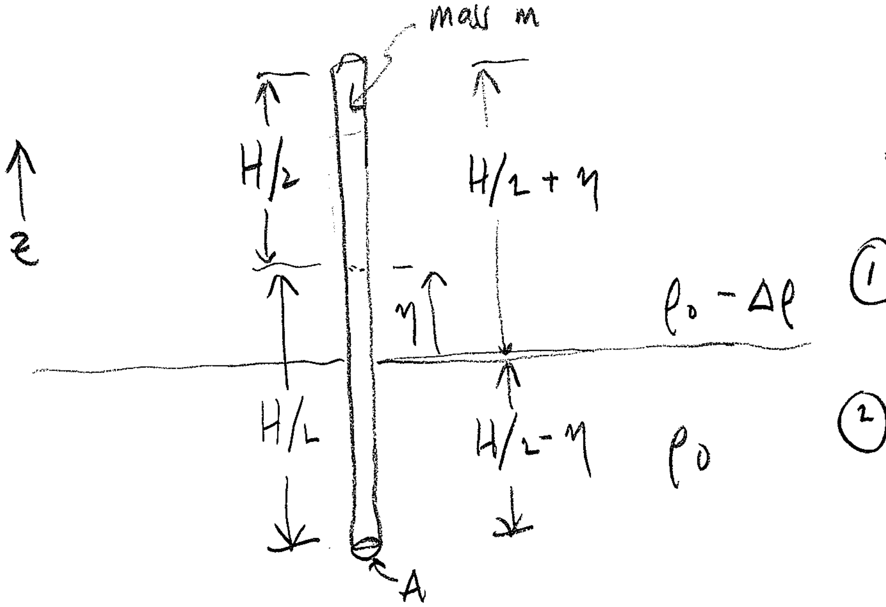


$t = 0$ $\quad$ $t = \frac{L}{U}$ $\quad$ $t \gg \frac{L}{U}$

2.iv.

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial z}\right) = \left[\underbrace{\frac{T_0}{L}}_{\text{constant}}, \underbrace{\frac{T_0}{H}\left(1 - \frac{U}{L} t\right)}_{\text{growing ever more negative in time (after } t = L/U)}\right]$$

3.i.



$AH = V$

$\underset{\sim}{F} = m \underset{\sim}{a}$

$$(+) \quad \underbrace{-mg}_{\text{gravity}} + \underbrace{A\left(\frac{H}{2}+\eta\right)(\rho-\Delta\rho)g + A\left(\frac{H}{2}-\eta\right)\rho_0 g}_{\text{Buoyancy} = g\int_V \rho \, dV} = m\frac{\partial^2 \eta}{\partial t^2}$$

Buoyancy $= g\int_V \rho \, dV$ ← ρ of the displaced fluid

for steady state $\eta = \eta_{tt} = 0$

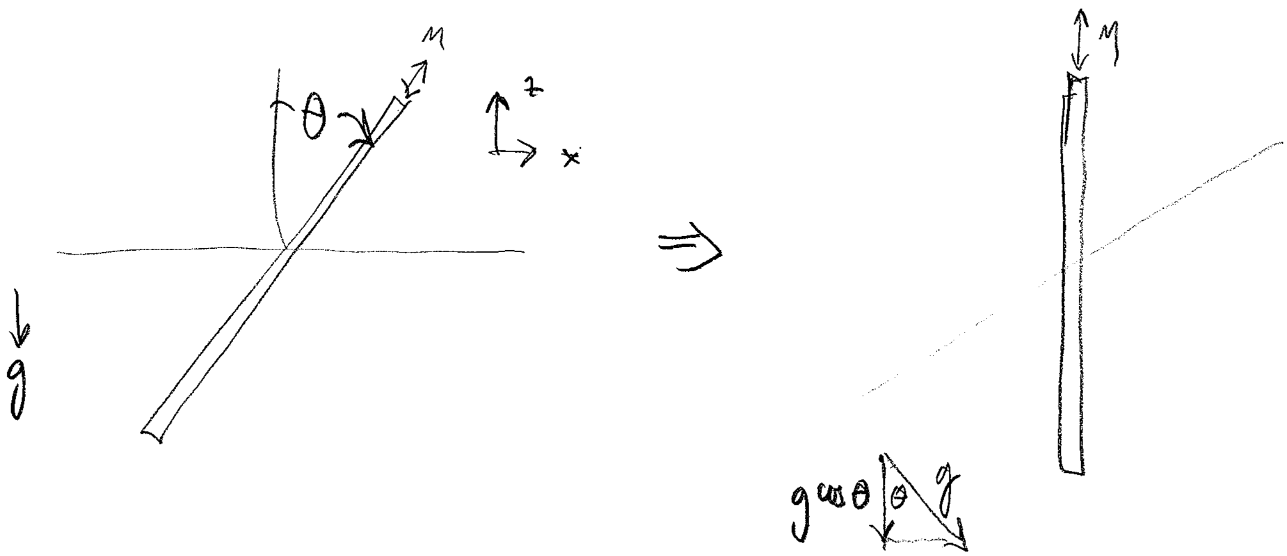
$$\Rightarrow \quad m = AH\rho_0 - \frac{AH}{2}\Delta\rho \cong AH\rho_0 \quad (\text{for } \Delta\rho \ll \rho_0)$$

+ then it is easy to simplify (+) to find

$$\boxed{\eta_{tt} + \frac{g'}{H}\eta = 0} \quad \text{where } g' \equiv \frac{g\Delta\rho}{\rho_0} \text{ is the "reduced gravity"}$$

$$\Rightarrow \eta = \eta_0 \cos \omega t \quad \text{where } \eta = \sqrt{\frac{g'}{H}}$$

3.ii. You can do this most easily by rotating your coordinate system



and the solution is the same as in 3.i.

except $\omega = \sqrt{\dfrac{g' \cos\theta}{H}}$

3.iii In layer 1 $m\eta_{tt} = (\rho_0 - \Delta\rho)Vg - mg$

and in 2 $m\eta_{tt} = \rho_0 V g - mg$

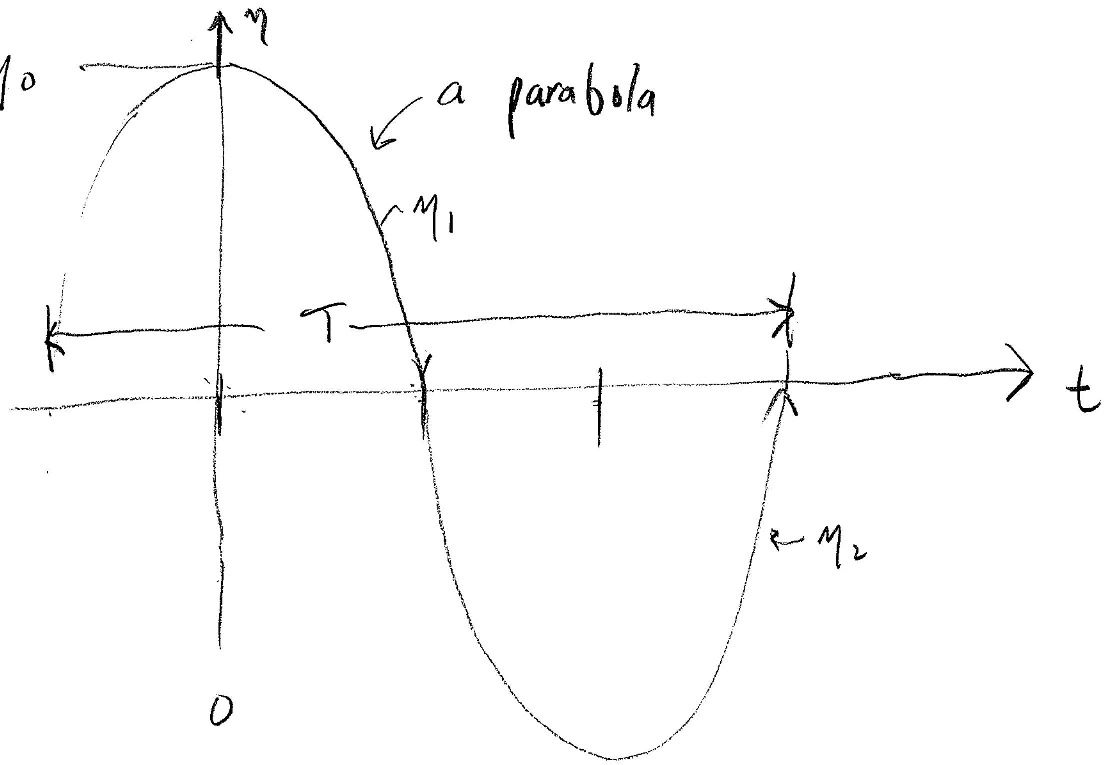
and $m = V\rho_0 - V\dfrac{\Delta\rho}{2}$

so the equations become

① $\eta_{tt} = -g'/2$

② $\eta_{tt} = g'/2$

which has solution



$$\left.\begin{aligned}\eta_1 &= \eta_0 - \frac{g'}{2}t^2 \\ \eta_2 &= \frac{g'}{2}\left(t - \frac{T}{2}\right)^2 - \eta_0\end{aligned}\right\}\ \text{and so on}$$

where $T = 4\sqrt{\dfrac{2\eta_0}{g'}} = 80$ sec. for $\eta_0 = 2$ m.