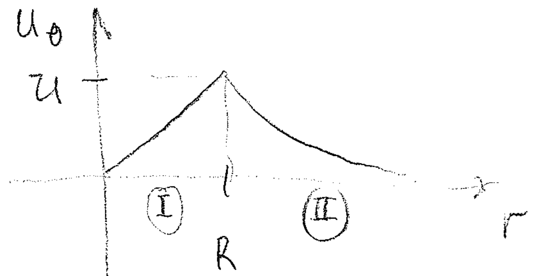
1.a $p(r)$ for a Rankine vortex



$p = p_\infty - P$ $\quad$ $p \to p_\infty$

r mom $\quad -\dfrac{u_\theta^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} \Rightarrow \dfrac{\partial p}{\partial r} = \rho\dfrac{u_\theta^2}{r}$

I $\boxed{u_\theta = \dfrac{\mathcal{U}}{R} r}$, $0 < r < R$, $\dfrac{\partial p}{\partial r} = \rho\dfrac{\mathcal{U}^2}{R^2} r \Rightarrow p = \frac{1}{2}\rho\mathcal{U}^2\dfrac{r^2}{R^2} + C$

II $\boxed{u_\theta = \mathcal{U} R r^{-1}}$, $R \le r$, $\dfrac{\partial p}{\partial r} = \rho \mathcal{U}^2 R^2 r^{-3} \Rightarrow p = -\frac{1}{2}\rho\mathcal{U}^2\dfrac{R^2}{r^2} + C$

$\therefore$ II $\quad p_\infty - p_{II} = \frac{1}{2}\rho\mathcal{U}^2\dfrac{R^2}{r^2}$ or $p_{II} = p_\infty - \frac{1}{2}\rho\mathcal{U}^2\dfrac{R^2}{r^2}$

or defining any $p_{II}(R) = p_\infty - \frac{1}{2}\rho\mathcal{U}^2$ so define $\boxed{P \equiv \frac{1}{2}\rho\mathcal{U}^2}$

$\therefore \boxed{p_{II} = p_\infty - P\dfrac{R^2}{r^2}}$ $\qquad \Rightarrow p(R) = p_\infty - P$

and I $\quad p(R) - p_I = \frac{1}{2}\rho\mathcal{U}^2 - \frac{1}{2}\rho\mathcal{U}^2\dfrac{r^2}{R^2}$

or $\quad p_\infty - P - p_I = P - P\dfrac{r^2}{R^2}$

$\Rightarrow \boxed{p_I = p_\infty - 2P + P\dfrac{r^2}{R^2}}$

Total pressure drop is $2p = \rho U^2 = p_\infty - p(0)$

- independent of R ✓
- half this at $r = R$ ✓

$\boxed{1.b}$ for $U = 90\ m\,s^{-1}$ and $\rho = 1.2\ kg\ m^{-3}$

$$\Delta p = \rho U^2 = 1.2 \times 90^2 = 9720\ Pa = .9 \times 10^4\ Pa$$

so this is 9% of $p_{ATM} \approx 10^5\ Pa$

$\boxed{1.c}$ for free surface flow $-\frac{1}{\rho}\frac{\partial p}{\partial r} = -g\frac{\partial \eta}{\partial r} \rightarrow \boxed{\eta = \frac{1}{\rho g}(p - p_\infty)}$

where we define $\eta \to 0$ as $r \to \infty$

so defining $E = \frac{p}{\rho g} = \frac{U^2}{2g}$

$\Rightarrow \boxed{\eta_{II} = -E\frac{R^2}{r^2}}$, $\boxed{\eta_I = -2E + E\frac{r^2}{R^2} = E\left(\frac{r^2}{R^2} - 2\right)}$

then $\boxed{KE_A = \frac{1}{2}\rho u_\theta^2 H \text{ and } APE_A = \frac{1}{2}\rho g \eta^2}$ ...

So for APE

$$\mathrm{APE} = \overset{\text{I}}{\int_0^R (\cancel{2}\pi r)\left(\tfrac{1}{\cancel{4}}\rho g\right) E^2 \left(\frac{r^4}{R^4} - 4\frac{r^2}{R^2} + 4\right) dr} + \overset{\text{II}}{\int_R^\infty (\cancel{2}\pi r)\left(\tfrac{1}{\cancel{4}}\rho g\right) E^2 \frac{R^4}{r^4}\, dr}$$

$$= \pi\rho g E^2 \left\{ \int_0^R \left(\frac{r^5}{R^4} - 4\frac{r^3}{R^2} + 4r\right) dr + \int_R^\infty R^4 r^{-3}\, dr \right\}$$

$$= \pi\rho g E^2 \left\{ \left[\frac{1}{6}\frac{1}{R^4} r^6 - \frac{1}{R^2} r^4 + 2r^2\right]_0^R - \left[\frac{1}{2} R^4 r^{-2}\right]_R^\infty \right\}$$

$$= \pi\rho g E^2 \left\{ \overbrace{\left(\frac{1}{6} - 1 + 2\right)}^{7/6} R^2 + \overbrace{\frac{1}{2}}^{3/6} R^2 \right\} \qquad \frac{7}{6} + \frac{3}{6} = \frac{10}{6} = \frac{5}{3}$$

$$\therefore \boxed{\mathrm{APE} = \frac{5}{3}\pi R^2 \rho g E^2}$$

units $L^2 \frac{M}{L^3} \frac{L}{T^2} L^2 : \frac{ML^2}{T^2}$ ✓ Joules and for KE

$$KE = \int_0^R (2\pi r)\,\tfrac{1}{2}\rho\,\frac{\mathcal{U}^2}{R^2} r^2 H\,dr + \int_R^\infty (2\pi r)\,\tfrac{1}{2}\rho\,\mathcal{U}^2 R^2 r^{-2} H\,dr$$

$$= \pi\rho\,\mathcal{U}^2 H \left\{ \int_0^R \frac{r^3}{R^2}\,dr + \int_R^\infty R^2 r^{-1}\,dr \right\}$$

$$= \pi\rho\,\mathcal{U}^2 H \left\{ \frac{1}{4}\frac{r^4}{R^2}\Big|_0^R + R^2 \ln r \Big|_R^\infty \right\}$$

$$= \pi\rho\,\mathcal{U}^2 H \left[\frac{R^2}{4} + R^2 \ln\left(\frac{\infty}{R}\right) \right]$$

$$\therefore \boxed{KE = \pi R^2 \rho\,\mathcal{U}^2 H \left[\frac{1}{4} + \ln\frac{\infty}{R} \right]}$$

dominated by this term, but in real world it is not a true $\frac{1}{r}$ velocity distribution to ∞

e.g. for a hurricane $R \simeq 30$ km

and note $\ln(10) = 2.3$, $\ln(100) = 4.6$

so KE is dominantly outside core, and total increases very slowly with radius.

$\boxed{2.a}$

$\boxed{x \text{ mm}}$ $(u_t = -g\eta_x - Ru)$ $(H \frac{\partial}{\partial x})$

$\boxed{\text{mass}}$ $(\eta_t + Hu_x = 0)$ $(\partial/\partial t)$

$$\Rightarrow \quad Hu_{xt} = -gH\eta_{xx} - RHu_x = -gH\eta_{xx} + R\eta_t$$

$$\eta_{tt} + Hu_{xt} = 0$$

$$\Rightarrow \quad \boxed{\eta_{tt} + R\eta_t - gH\eta_{xx} = 0}$$

$\boxed{2.b}$ using $\eta = \eta_0 \exp i(kx - \omega t)$

$$\Rightarrow \quad -\omega^2 - i\omega R + gHk^2 = 0$$

$$\text{or} \quad \boxed{\omega^2 + iR\omega - gHk^2 = 0}$$

solve for roots

$$\omega = \frac{-iR \pm \left[-R^2 + 4gHk^2\right]^{\frac{1}{2}}}{2} = \boxed{\underbrace{\pm\left[gHk^2 - \left(\frac{R}{2}\right)^2\right]^{\frac{1}{2}}}_{\omega_R} - i\left(\frac{R}{2}\right) = \omega_R + i\omega_I}$$

= dispersion relation

real assuming $gHk^2 > \left(\frac{R}{2}\right)^2$

so we may write the solutions as

$$\eta = \eta_0 \cos(kx - \omega_R t)\, e^{-\frac{R}{2}t}$$

$\boxed{2.c}$ the waves are dispersive because $\dfrac{\omega_R}{k}$ is a function of k

$$c_p = \frac{\omega_R}{k} = \pm\left[gH - \left(\frac{R}{2k}\right)^2 \right]^{\frac{1}{2}}$$

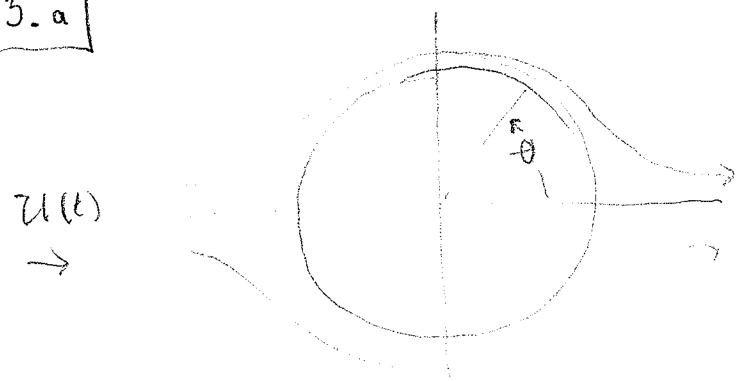
$\boxed{2.d}$ solving for λ, given ω_R

$$\omega_R^2 = gHk^2 - \left(\frac{R}{2}\right)^2 \Rightarrow gHk^2 = \omega_R^2 - \left(\frac{R}{2}\right)^2$$

$$k = \left[\frac{\omega_R^2 - (R/2)^2}{gH} \right]^{\frac{1}{2}} = \frac{2\pi}{\lambda} \qquad \text{(take positive root)}$$

$$\Rightarrow \lambda = 2\pi \frac{\sqrt{gH}}{\left[\omega_R^2 - (R/2)^2\right]^{\frac{1}{2}}} \simeq 2000\,\text{km}$$

$\boxed{3.a}$



$$\varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta$$

generalized Bernoulli is

$$\rho\frac{\partial\varphi}{\partial t} + \frac{1}{2}\rho\,\underline{u}\cdot\underline{u} + p = C(t)$$

$$\Rightarrow p = C - \frac{1}{2}\rho\left(u_\theta^2 + u_r^2\right) - \rho\frac{\partial\varphi}{\partial t}$$

at $r = a$ $u_r = 0$ and $\varphi = 2Ua\cos\theta$

typo: should be u_theta = ...

$$u_r = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{1}{\cancel{a}}(-2U\cancel{a})\sin\theta = -2U\sin\theta$$

typo: should be u_theta^2 = ...

$$u_r^2 = 4U^2\sin^2\theta$$

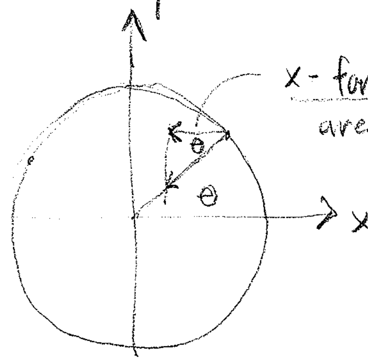
so

$$p(r=a) = C - \frac{1}{2}\rho\, 4U^2\sin^2\theta - \rho\, 2\frac{\partial U}{\partial t}a\cos\theta$$

or

$$\boxed{p(r=a) = C - 2\rho U^2\sin^2\theta - 2\rho a\frac{\partial U}{\partial t}\cos\theta}$$

3.b



$\frac{x\text{-force}}{\text{area}} = -p(r=a)\cos\theta$

$da = a\,d\theta$

$$\Rightarrow \quad \frac{x \text{ Force}}{\text{unit length of cylinder}} = \int_0^{2\pi} -p(r=a)\cos\theta \; a \, d\theta$$

$$\frac{F^x}{\text{unit } z} = \int_0^{2\pi} \left[-a\cos\theta \; C + 2\rho U^2 a \sin^2\theta \cos\theta + 2\rho a^2 \frac{\partial U}{\partial t} \cos^2\theta \right] d\theta$$

$\int al = 0$ $\qquad$ $\int al = 0$ $\qquad$ $\int al = \pi$

so $\boxed{\frac{F^x}{\text{unit } z} = 2\pi a^2 \rho \frac{\partial U}{\partial t}} = 2\rho A \frac{\partial U}{\partial t}$ $\quad$ where $A = \pi a^2$

force to $\rightarrow$ when U_t is positive