Operations Research Letters 39 (2011) 144–149

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Online lot-sizing problems with ordering, holding and shortage costs

Michael R. Wagner *

Graduate Business Programs, Saint Mary's College of California, Moraga, CA 94556, United States

ARTICLE INFO

Article history:
Received 22 July 2010
Accepted 16 December 2010
Available online 4 February 2011

Keywords:
Online optimization
Inventory management
Duality theory

ABSTRACT

We study inventory management problems where demands are revealed incrementally and procurement decisions must be made before the demands are realized. There are no probabilistic distributions nor non-trivial bounds to characterize demands. We consider two cost minimization problems: (1) perishable products with lost sales and (2) durable products with backlogged demand. In both problems, costs are period dependent. These problems are analyzed by utilizing linear-fractional programming and duality theory. Structural results are proved and then developed into practical strategies.

1. Introduction

This paper presents a systematic analysis of two general inventory control problems where non-negative demands are revealed incrementally, after procurement decisions have been made. No distributions or non-trivial bounds are available to characterize demands. The first problem considers perishable products with lost sales and the second considers durable products with backlogged demand. We utilize *competitive analysis*, a (traditional) computer-science approach for evaluating algorithms functioning under incomplete information. Furthermore, competitive analysis is based on the worst-case *competitive ratio* metric, which makes the procurement strategies naturally risk averse. For other examples of risk-averse decision making in inventory management, see [8,9,17,22] and the references therein.

Inventory management models serve as building blocks for more sophisticated Material Requirements Planning (MRP) and Enterprise Resource Planning (ERP) systems, such as those offered by SAP and Oracle. There is criticism of these systems due to their "nervousness", or the sensitivity of the solution to demand. Therefore, studying inventory management models from a demandless perspective can provide knowledge to improve the stability of these systems. In particular, our analysis results in simple strategies, with provable performance guarantees, that do not require knowledge of current or future demand.

1.1. Related literature

There is limited literature that considers inventory management when demand is unknown, without any defining characteristics. Levi [7] considers the well-known Joint Replenishment Problem (JRP) when there is no probabilistic distribution or additional information to characterize unknown demand. Levi et al. [18] design a sampling-based procedure to overcome the absence of distributions and [11] design an algorithm that directly estimates value functions without the need for demand distributions. Huh, Huh and Janakiraman [14,12] and Huh and Levi [13] study various inventory management problems where a demand distribution is lacking, but sales data are available. Additionally, [19] compare distribution-free and stochastic algorithms for the multi-period Newsvendor Problem. In the related area of revenue management, Ball and Queyranne [4] consider airfare booking when there is no distribution for the stream of customers purchasing airfare; [16] analyze the case where the lower and upper bounds for customer demand exist and [10] consider distribution-free methods for booking control. Lan et al. [15] consider overbooking and fare-class allocation when there is limited demand information and [3] consider distribution-free methods in revenue management and competition.

Most related to our paper is the recent work of [21], which builds upon the work of [1,2,5,20]; these latter works all analyzed approximation algorithms with constant performance guarantees for variants of the dynamic lot-sizing problem. Van den Heuvel and Wagelmans [21] specifically consider *online* algorithms for a dynamic lot-sizing problem and are able to prove constant performance bounds. However, these approaches differ significantly from the approach we take in our paper due to the following: (1) These references do not allow shortages and our paper considers product shortages (intentional or not) as a fundamental characteristic; indeed, we prove lower bounds, on the performance guarantee *of any algorithm* for our models, that depend on the shortage costs. In addition, it is precisely the addition of shortages that precludes any constant competitive ratio bound. (2) In the online approach of [21], the period k demand is available when making a procurement decision for period k; in

* Corresponding address: 380 Moraga Road, Moraga, CA 94556, United States.
E-mail address: mrw2@stmarys-ca.edu.

0167-6377/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.orl.2011.01.007

our approach, this demand is not available and is only revealed *after* a procurement decision is made, which implies that shortages cannot be eliminated. (3) The above approaches do not allow demand backlogging and allow inventory holding in all periods; in our approach, we either (a) disallow both backlogging and inventory carryover (i.e., perishable products) or (b) allow both. (4) We design and analyze in depth a Make-to-Order strategy that never maintains positive inventory — this strategy is beyond the scope of these related references. (5) In contrast to the online approach of [21], our costs are period dependent and can vary.

1.2. Contributions

The first problem that we study consists of designing the procurement strategy for a single *perishable* product over a finite planning horizon with period-dependent costs. If demand exceeds product availability in any period, the excess demand is lost. We characterize the competitive ratio of an arbitrary (albeit intelligent) procurement strategy. The second problem that we study consists of designing the procurement and inventory management strategy for a single *durable* product, that can be inventoried, over a finite planning horizon with period-dependent costs. Additionally, excess demand is backlogged for future periods. The main result of this part of the paper is a two-part structural result where we (1) derive a sufficient condition for the existence of a finite competitive ratio and (2) provide lower and upper bounds for the competitive ratio. We then provide an in-depth example of utilizing these structural results to design and analyze a Make-to-Order strategy that identifies the best times to fulfill backlogged demand. We characterize the competitive ratio of this specific strategy through lower and upper bounds; if ordering and shortage costs are identical over all periods, the lower and upper bounds coincide and we know the exact value of the competitive ratio. Finally, we contrast our paper with [23], which introduces unit revenues to the models analyzed in this paper. Maximizing profit, rather than minimizing cost, renders the usual competitive ratio metric ill-posed, since profits can be positive or negative, and [23] introduces and utilizes a new worst-case metric, different from the competitive ratio that is appropriate for a profit objective. However, simply setting revenues to zero in [23] will not give results remotely similar to this paper; in fact, by setting revenues to zero, [23] suggests ordering negative quantities of products. Therefore, the cost minimization and profit maximization objectives in the context of online inventory management are fundamentally different and warrant different studies.

Outline of the paper

In Section 2 we explain our notation, state our assumptions, derive the two models we study and discuss competitive analysis. In Section 3 we analyze the perishable product with lost sales model and in Section 4 we analyze the durable product with backlogged demand model, concluding with the analysis of a specific Make-to-Order strategy.

2. Preliminaries

2.1. Notation

We begin by explaining the notation that we utilize throughout the paper. Scalar values are represented in regular type and vectors are represented in boldface type. For example, $\mathbf{x} = (x_1, \ldots, x_n)$ is a column vector of n elements. $\mathbf{x}'$ is the transpose (row) vector of the (column) vector $\mathbf{x}$. Additionally, we define $\mathbf{e} = (1, \ldots, 1)$ as the n-dimensional vector of all ones. Also, $\delta(x)$ is the indicator function; i.e., $\delta(x) = 1$ if $x > 0$ and $\delta(x) = 0$ if $x = 0$. Finally, let $\mathbf{x}^+ = \max\{\mathbf{x}, \mathbf{0}\}$ and $\mathbf{x}^- = \max\{-\mathbf{x}, \mathbf{0}\}$, where the max and min operators are defined component-wise. Next, we provide a useful result that links a specific linear-fractional program with linear programming.

Lemma 2.1 (*Linear-Fractional Programming*)**.** *If* $\{\mathbf{x} : \mathbf{f}'\mathbf{x} + g > 0,\ \mathbf{x} \geq \mathbf{0}\}$ *is non-empty, then the optimization problems*

$$\begin{array}{ll} \max\limits_{\mathbf{x}} & \dfrac{\mathbf{c}'\mathbf{x} + d}{\mathbf{f}'\mathbf{x} + g} \\ \text{s.t.} & \mathbf{f}'\mathbf{x} + g > 0 \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \text{and} \quad \begin{array}{ll} \max\limits_{\mathbf{y}, z} & \mathbf{c}'\mathbf{y} + dz \\ \text{s.t.} & \mathbf{f}'\mathbf{y} + gz = 1 \\ & \mathbf{y} \geq \mathbf{0},\ z \geq 0 \end{array}$$

are equivalent.

Proof. A proof can be found in [6] on page 151. □

2.2. Model derivation

We begin by detailing the data for our problems. We consider the n-period inventory management of a single product where the objective is to minimize total cost. In period i, $d_i \geq 0$ is the demand for the product, $q_i \geq 0$ is the ordering quantity, $c_i \geq 0$ is the unit ordering cost, $K_i \geq 0$ is the fixed ordering cost for placing an order, $h_i \geq 0$ is the unit inventory holding cost and $s_i \geq 0$ is the unit inventory shortage cost, which will either represent a lost sales cost or a backlogging cost (which will be clear from the context). In vector notation, the parameters are $\mathbf{d}, \mathbf{q}, \mathbf{c}, \mathbf{K}, \mathbf{h}, \mathbf{s} \geq \mathbf{0}$. We make three assumptions that hold throughout and then derive the models.

Assumption 2.2. $\mathbf{c}, \mathbf{K}, \mathbf{h}, \mathbf{s} > \mathbf{0}$.

Assumption 2.3. $\mathbf{d} \neq \mathbf{0}$.

Assumption 2.4. There exists a period in which it is optimal to procure a positive quantity.

2.2.1. Perishable products with lost sales

We begin with the case where the product is perishable (e.g., certain food products) and cannot be inventoried for future periods. Additionally, we assume that any unmet demand is lost forever. The inventory holding costs h_i have the managerial interpretation of a write-off (of perished inventory) cost and the inventory shortage costs s_i have the interpretation of quantifying lost sales (e.g., lost revenues, etc.). In period i, the write-off cost is $h_i(q_i - d_i)^+$, the shortage cost is $s_i(d_i - q_i)^+$ and the ordering costs are $c_i q_i + K_i \delta(q_i)$. The cost minimization model in this case is

$$\min_{\mathbf{q} \geq \mathbf{0}} \sum_{i=1}^{n} (c_i q_i + h_i(q_i - d_i)^+ + s_i(d_i - q_i)^+ + K_i \delta(q_i)). \tag{1}$$

2.2.2. Durable products with backlogging

The second framework we study is one where the product is not perishable and can be inventoried to satisfy demand in later periods. Additionally, if demand exceeds available inventory, it is backlogged and satisfied in future periods.

We define the inventory I_i *at the end* of period i, which must satisfy the inventory balance constraints $I_i = I_{i-1} + q_i - d_i$, for $i = 1, \ldots, n$, with initial inventory $I_0 = 0$. We next explicitly define positive and negative inventory as $I_i^+ = \max\{I_i, 0\}$ and $I_i^- = \max\{-I_i, 0\}$, respectively. Clearly, $I_i = I_i^+ - I_i^-$ and $|I_i| = I_i^+ + I_i^-$. Finally, note that $I_i = \sum_{j=1}^{i}(q_j - d_j)$; this identity is utilized in subsequent proofs.

In period i, the inventory holding cost is $h_i I_i^+$, the inventory shortage cost is $s_i I_i^-$ and the inventory ordering cost is $K_i \delta(q_i) +$

$c_i q_i$, where δ is the indicator function. In summary, the cost minimization model in this case is

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} (c_i q_i + h_i I_i^+ + s_i I_i^- + K_i \delta(q_i)) \\ \text{s.t.} \quad & I_i = I_{i-1} + q_i - d_i, \quad i = 1, \ldots, n, \; I_0 = 0 \\ & q_i \geq 0, \; i = 1, \ldots, n. \end{aligned} \tag{2}$$

If demands were known deterministically, our model would be that which was analyzed in [25]. If we add the constraint $I_i \geq 0$ for all i, the model is the well-known Wagner–Whitin Model, which was originally analyzed in [24].

2.3. Competitive analysis: online and offline problems

In this paper we study inventory management decisions without knowing customer demands. In particular, in period i, an *online player* must decide how much product to order q_i without knowing the demand d_i. However, this online player does know the cost structure of all periods $j \leq i$, as well as all previous demands d_j and previous decisions q_j for $j < i$. In particular, in period i, c_j, K_j, s_j, h_j for $j \leq i$ are known. Additionally, the online player does not know how many periods n there are in the planning horizon. The cost that an online player accrues over n periods will be denoted as $Z(\mathbf{d})$. An *offline adversary* knows all demands $\mathbf{d}$ a priori and makes optimal decisions. The cost that an offline adversary accrues over n periods will be denoted as $Z^*(\mathbf{d})$. In particular, $Z^*(\mathbf{d})$ is defined as the optimal solution of either Model (1) or Model (2) — the relevant model will be clear from context. Clearly $Z^*(\mathbf{d}) \leq Z(\mathbf{d})$. It will also be convenient to write $Z(\mathbf{d}) = \sum_{i=1}^{n} Z_i(d_i)$ where $Z_i(d_i)$ is the cost contribution in period i; similarly, $Z^*(\mathbf{d}) = \sum_{i=1}^{n} Z_i^*(d_i)$. Traditionally, the quality of an online strategy is measured via the *competitive ratio*. In our notation, the competitive ratio is defined as the smallest value of $\alpha \geq 1$ such that

$$Z(\mathbf{d}) \leq \alpha Z^*(\mathbf{d}) + \beta, \quad \forall \mathbf{d} \geq \mathbf{0}, \tag{3}$$

where β is a demand-independent constant. If $\beta \leq 0$, we say the competitive ratio α is *strict*. Finally, Assumptions 2.2–2.4 imply that $Z^*(\mathbf{d}) > 0$ and an equivalent definition of a *strict* competitive ratio would be

$$\alpha = \sup_{\substack{\mathbf{d} \geq \mathbf{0} \\ \mathbf{d} \neq \mathbf{0}}} \frac{Z(\mathbf{d})}{Z^*(\mathbf{d})};$$

a strict competitive ratio is clearly preferable to one that is not. All but one of the competitive ratio results in this paper are strict.

3. Perishable products with lost sales

In this section we prove the following theorem by first considering a single-period model and then extending the analysis to a finite planning horizon; note that Assumption 2.4 implies that there exists at least one period i where $c_i < s_i$ (otherwise it is optimal to never order anything).

Theorem 3.1. *In period i, if $c_i \geq s_i$, order $q_i = 0$ units and if $c_i < s_i$, order q_i units. The competitive ratio of this strategy is at most*

$$\max_{i: c_i < s_i} \left\{ \max \left\{ \left(\frac{c_i + h_i}{K_i} \right) q_i + 1, \frac{s_i}{c_i} \right\} \right\}.$$

Furthermore, the competitive ratio of any algorithm is at least $\max_{i: c_i < s_i} \{s_i / c_i\}$.

Corollary 3.2. *In period i, if $c_i \geq s_i$, order $q_i = 0$ units and if $c_i < s_i$, order $q_i = K_i(s_i - c_i)/(c_i(c_i + h_i))$. The competitive ratio of this strategy is exactly* $\max_{i: c_i < s_i} \{s_i / c_i\}$.

3.1. A single-period model

In this section we consider the special case of a single period with unknown demand d. The specific model is

$$\min_{q \geq 0} (cq + h(q - d)^+ + s(d - q)^+ + K\delta(q)).$$

Note that Assumption 2.4 implies that $c < s$; otherwise, it is optimal for both the online and offline players to order nothing. We include this observation in the following theorem for completeness. Finally, note that for a single period, the competitive ratio is *strict* and best possible.

Theorem 3.3. (1) *If $c \geq s$, it is optimal (i.e., the strict competitive ratio $= 1$) to order zero units.*
(2) *If $c < s$, the strict competitive ratio of ordering q units is equal to*

$$\max \left\{ \left(\frac{c + h}{K} \right) q + 1, \frac{s}{c} \right\}.$$

Proof. As a function of d, the optimal offline adversarial solution is to either order $q = 0$ or $q = d$, which implies $Z^*(d) = \min\{sd, cd + K\}$. If $c \geq s$, it is optimal for the offline adversary to order zero units; since the online player knows the cost structure (i.e., the values of c and s), it also orders zero units, which implies that the online and offline costs are identical and the competitive ratio is one. The remainder of the proof considers the case $c < s$. Note that we have the following lower bound on the competitive ratio

$$\rho = \sup_{d \geq 0} \left(\frac{Z(d)}{Z^*(d)} \right) \geq \sup_{d \geq 0} \left(\frac{Z(d)}{cd + K} \right).$$

Next, we note that Assumption 2.4 implies that $cd + K < sd$, which induces the strict lower bound $d > K/(s - c)$. The competitive ratio is therefore equal to the first term below and bounded from above by the second term

$$\rho = \sup_{d > K/(s-c)} \left(\frac{Z(d)}{cd + K} \right) \leq \sup_{d \geq 0} \left(\frac{Z(d)}{cd + K} \right).$$

Consequently, the competitive ratio is exactly $\sup_{d \geq 0}(Z(d)/(cd + K))$. We consider two possible cases that determine the structure of $Z(d)$, where q is now the online player's procurement quantity. We assume $q > 0$ and consider $q = 0$ subsequently.

Case 1 ($q > d$). We have that

$$\rho = \sup_{d \geq 0} \left(\frac{-hd + (c + h)q + K}{cd + K} \right),$$

the right-hand side of which is a linear-fractional program, which, applying Lemma 2.1, can be written as the linear program

$$\begin{aligned} \max_{y,z} \quad & -hy + ((c + h)q + K)z \\ \text{s.t.} \quad & cy + Kz = 1 \qquad\qquad \text{whose dual is} \\ & y, z \geq 0. \end{aligned}$$

$$\begin{aligned} \min_{\alpha} \quad & \alpha \\ \text{s.t.} \quad & \alpha c \geq -h \\ & \alpha K \geq (c + h)q + K. \end{aligned}$$

The competitive ratio is equal to the optimal dual solution $q(c + h)/K + 1$.

Case 2 ($q \leq d$). We have that

$$\rho = \sup_{d \geq 0} \left(\frac{sd + (c - s)q + K}{cd + K} \right),$$

the right-hand side of which is a linear-fractional program, which can be written as the linear program

$$\begin{aligned} \max_{y,z}\quad & sy + (K-(s-c)q)z \\ \text{s.t.}\quad & cy + Kz = 1 \\ & y, z \geq 0. \end{aligned} \qquad \text{whose dual is}$$

$$\begin{aligned} \min_{\alpha}\quad & \alpha \\ \text{s.t.}\quad & \alpha c \geq s \\ & \alpha K \geq K-(s-c)q. \end{aligned}$$

The competitive ratio is equal to the optimal dual solution $\max\{s/c, 1-q(s-c)/K\} = s/c$.

Finally, if $q = 0$, we set $K-(s-c)q = 0$ and the dual in Case 2 simplifies to $\min\{\alpha : \alpha \geq s/c\}$, which implies a competitive ratio equal to s/c.

Since a priori an online player does not know if Case 1 or Case 2 will occur, the competitive ratio is calculated as the maximum of the two dual solutions: $\max\{q(c+h)/K+1, s/c\}$. □

3.2. *Proof of Theorem 3.1*

Note that Theorem 3.1 is not an immediate consequence of Theorem 3.3 since periods with zero demand and periods where the offline player orders nothing must now be carefully analyzed.

Proof. Recall that $Z(\mathbf{d}) = \sum_{i=1}^n Z_i(d_i)$, $Z^*(\mathbf{d}) = \sum_{i=1}^n Z_i^*(d_i)$ and define $S = \{i : c_i < s_i\}$. Assumption 2.4 implies that $S \neq \emptyset$. We partition S into $S_1 = S \cap \{i : d_i > K_i/(s_i-c_i)\}$ and $S_2 = S \cap \{i : d_i \leq K_i/(s_i-c_i)\}$ since the offline adversary orders d_i units for $i \in S_1$ and orders 0 units for $i \in S_2$; Assumption 2.4 also implies that $S_1 \neq \emptyset$. The proof of Theorem 3.3 implies that, for all $\mathbf{d} = (d_1, \ldots, d_n)$,

$$Z_i(d_i) \leq \max\left\{\left(\frac{c_i+h_i}{K_i}\right)q_i + 1, \frac{s_i}{c_i}\right\} Z_i^*(d_i), \quad i \in S_1 \quad \text{and}$$

$$Z_i(d_i) = Z_i^*(d_i), \quad i \notin S.$$

For $i \in S_2$, we have that $Z_i(d_i) = K_i\delta(q_i) + c_iq_i + h_i(q_i-d_i)^+ + s_i(d_i-q_i)^+$; if $q_i \geq d_i$,

$$\begin{aligned} Z_i(d_i) &= K_i\delta(q_i) + c_iq_i + h_i(q_i-d_i) \\ &\leq K_i\delta(q_i) + c_iq_i + h_iq_i \triangleq \gamma_{i1} \end{aligned}$$

and if $q_i < d_i$,

$$\begin{aligned} Z_i(d_i) &= K_i\delta(q_i) + c_iq_i + s_i(d_i-q_i) \\ &\leq K_i\delta(q_i) + c_iq_i + s_i(K_i/(s_i-c_i)-q_i) \triangleq \gamma_{i2}. \end{aligned}$$

Clearly, for $i \in S_2$, $Z_i(d_i) \leq \max\{\gamma_{i1}, \gamma_{i2}\} \triangleq \gamma_i$. Defining $\gamma = \sum_{i \in S_2} \gamma_i$, a *demand-independent constant*, we have that

$$\begin{aligned} Z(\mathbf{d}) &= \sum_{i\in S_1} Z_i(d_i) + \sum_{i\notin S} Z_i(d_i) + \sum_{i \in S_2} Z_i(d_i) \\ &\leq \sum_{i\in S_1} \max\left\{\left(\frac{c_i+h_i}{K_i}\right)q_i + 1, \frac{s_i}{c_i}\right\} Z_i^*(d_i) \\ &\quad + \sum_{i\notin S} Z_i^*(d_i) + \gamma \\ &\leq \max_{i\in S}\left\{\left(\frac{c_i+h_i}{K_i}\right)q_i + 1, \frac{s_i}{c_i}\right\} Z^*(\mathbf{d}) + \gamma. \end{aligned}$$

We next give a lower bound for the competitive ratio. Arbitrarily choose $j \in S$ and let $d_i = 0$ for all $i \neq j$. Therefore, $Z^*(\mathbf{d}) = Z_j^*(d_j)$ and $Z(\mathbf{d}) \geq Z_j(d_j)$. We only consider $d_j > K_j/(s_j-c_j)$, so that the offline adversary orders d_j, which implies $Z_j^*(d_j) = c_jd_j + K_j$. If the online player orders q_j in period j, we then consider $d_j \geq q_j$, which implies $Z_j(d_j) = c_jq_j+K_j+s_j(d_j-q_j)$. As $d_j \to \infty$, the ratio of $Z_j(d_j)$ to $Z_j^*(d_j)$ approaches s_j/c_j. Consequently, as $d_j \to \infty$, $Z(\mathbf{d})/Z^*(\mathbf{d}) \geq s_j/c_j$; since $j \in S$ was chosen arbitrarily, we have proven the claimed lower bound. □

4. Durable products with backlogged demand

In this section, we study online procurement strategies for durable products with backlogged demand. Let $P = \{i : I_i \geq 0\}$ and $N = \{i : I_i \leq 0\}$ denote the periods of non-negative and non-positive inventory, respectively. If inventory is zero in period i, we can assign i arbitrarily to P or N. Clearly P and N are subsets of $\{1, \ldots, n\}$, so if $i \notin \{1, \ldots, n\}$, then $i \notin P$ and $i \notin N$. We begin with the following linear-combination characterization of the cost of an *arbitrary* online procurement strategy. Note that a summation over an empty set is defined as zero.

Lemma 4.1. *For an arbitrary online strategy $\mathbf{q} \geq \mathbf{0}$, we can write the online cost as*

$$Z(\mathbf{d}) = \mathbf{a}'\mathbf{d} + \mathbf{b}'\mathbf{q} + \mathcal{K},$$

where $a_i = \sum_{j=i, j\in N}^n s_j - \sum_{j=i, j\in P}^n h_j$, $b_i = c_i + \sum_{j=i, j\in P}^n h_j - \sum_{j=i, j\in N}^n s_j$, *for* $i = 1, \ldots, n$ *and* $\mathcal{K} = \sum_{i=1}^n K_i\delta(q_i)$.

Proof. Decomposing along the partition (P, N) and noting that $I_i^+ = I_i$ for $i \in P$ and $I_i^- = -I_i$ for $i \in N$, we have that the online cost

$$\begin{aligned} Z(\mathbf{d}) &= \mathbf{c}'\mathbf{q} + \sum_{i\in P} h_iI_i - \sum_{i\in N} s_iI_i + \mathcal{K} \\ &= \mathbf{c}'\mathbf{q} + \sum_{i\in P} h_i \sum_{j=1}^i (q_j-d_j) - \sum_{i\in N} s_i \sum_{j=1}^i (q_j-d_j) + \mathcal{K} \\ &= \mathbf{c}'\mathbf{q} + \sum_{i=1}^n (q_i-d_i) \sum_{\substack{j=i \\ j\in P}}^n h_j - \sum_{i=1}^n (q_i-d_i) \sum_{\substack{j=i \\ j\in N}}^n s_j + \mathcal{K}, \end{aligned}$$

where the third equality is obtained by inverting the order of the summations. Letting $a_i = \sum_{j=i, j\in N}^n s_j - \sum_{j=i, j\in P}^n h_j$ and $b_i = c_i + \sum_{j=i, j\in P}^n h_j - \sum_{j=i, j\in N}^n s_j$ completes the proof. □

Lower and upper bounds for the optimal offline cost are given in the next lemma.

Lemma 4.2. *The optimal offline cost of Model (2) has the following lower and upper bounds*

$$\boldsymbol{\alpha}'\mathbf{d} \leq Z^*(\mathbf{d}) \leq \mathbf{c}'\mathbf{d} + \mathbf{K}'\mathbf{e}, \quad \forall \mathbf{d} \geq \mathbf{0},$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is defined as

$$\begin{aligned} \alpha_i = \min\Bigg\{ & \min_{1\leq j\leq i-1}\left\{c_j + \sum_{k=j}^{i-1} h_k\right\}, c_i, \\ & \min_{i+1\leq j\leq n}\left\{c_j + \sum_{k=i}^{j-1} s_k\right\}, \sum_{k=i}^n s_k\Bigg\}, \quad i = 1, \ldots, n \end{aligned}$$

and can be interpreted as the minimum marginal cost of satisfying demand d_i by considering (1) *procuring in period $j < i$ and carrying the inventory to period i,* (2) *procuring in period i,* (3) *backlogging until period $j > i$, or* (4) *the cost of not satisfying the demand.*

Proof. Since ordering d_i in period i is a feasible offline solution, we have the upper bound $Z^*(\mathbf{d}) \leq \mathbf{c}'\mathbf{d} + \mathbf{K}'\mathbf{e}$. To begin deriving a valuable lower bound, we remove fixed ordering costs and we see that

$$\begin{aligned} Z^*(\mathbf{d}) &\geq \min_{\mathbf{q}\geq\mathbf{0}} \sum_{i=1}^n (c_iq_i + s_iI_i^- + h_iI_i^+) \\ &= \min_{\mathbf{q}\geq\mathbf{0}} \sum_{i=1}^n \left(c_iq_i + s_i \max\left\{\sum_{j=1}^i (d_j-q_j), 0\right\}\right. \\ &\quad \left. + \; h_i \max\left\{\sum_{j=1}^i (q_j-d_j), 0\right\}\right). \end{aligned} \tag{4}$$

By introducing additional variables $p_i, r_i \geq 0,\ i = 1, \ldots, n$, and noting that $\mathbf{s}, \mathbf{h} > \mathbf{0}$, we rewrite the Lower Bound (4) as the linear program

$$\begin{aligned} \min_{\mathbf{p},\mathbf{q},\mathbf{r}} \quad & \mathbf{s}'\mathbf{p} + \mathbf{c}'\mathbf{q} + \mathbf{h}'\mathbf{r} \\ \text{s.t.} \quad & p_i + \sum_{j=1}^{i} q_j = \sum_{j=1}^{i} d_j, \quad i = 1, \ldots, n \\ & r_i - \sum_{j=1}^{i} q_j = -\sum_{j=1}^{i} d_j, \quad i = 1, \ldots, n \\ & \mathbf{p}, \mathbf{q}, \mathbf{r} \geq \mathbf{0} \end{aligned} \quad \text{or equivalently}$$

$$\begin{aligned} \min_{\mathbf{p},\mathbf{q},\mathbf{r}} \quad & \mathbf{s}'\mathbf{p} + \mathbf{c}'\mathbf{q} + \mathbf{h}'\mathbf{r} \\ \text{s.t.} \quad & p_1 + q_1 = d_1 \\ & p_i - p_{i-1} + q_i = d_i, \quad i = 2, \ldots, n \\ & r_1 - q_1 = -d_1 \\ & r_i - r_{i-1} - q_i = -d_i, \quad i = 2, \ldots, n \\ & \mathbf{p}, \mathbf{q}, \mathbf{r} \geq \mathbf{0}. \end{aligned}$$

Letting $\gamma_i, \beta_i,\ i = 1, \ldots, n$, be dual variables, the dual linear program can be written as

$$\begin{aligned} \max_{\boldsymbol{\gamma},\boldsymbol{\beta}} \quad & (\boldsymbol{\gamma} + \boldsymbol{\beta})'\mathbf{d} \\ \text{s.t.} \quad & \gamma_i \leq s_i + \gamma_{i+1}, \quad i = 1, \ldots, n-1 \\ & \gamma_n \leq s_n \\ & \beta_{i+1} \leq h_i + \beta_i, \quad i = 1, \ldots, n-1 \\ & 0 \leq h_n + \beta_n \\ & \gamma_i + \beta_i \leq c_i, \quad i = 1, \ldots, n. \end{aligned}$$

Now, if $\boldsymbol{\beta} = \mathbf{0}$, the recursion $\gamma_i = \min\{c_i, s_i + \gamma_{i+1}\}$, with base case $\gamma_n = \min\{c_n, s_n\}$, defines a feasible solution to the dual; note that $\gamma_i = \min\{c_i, \min_{i+1 \leq j \leq n}\{c_j + \sum_{k=i}^{j-1} s_k\}, \sum_{k=i}^{n} s_k\},\ i = 1, \ldots, n-1$ satisfies this recursion. Alternatively, if $\boldsymbol{\gamma} = \mathbf{0}$, the recursion $\beta_{i+1} = \min\{c_{i+1}, h_i + \beta_i\}$, with base case $\beta_1 = c_1$, defines another feasible solution to the dual; note that $\beta_i = \min\{c_i, \min_{1 \leq j \leq i-1}\{c_j + \sum_{k=j}^{i-1} h_k\}\},\ i = 2, \ldots, n$ satisfies this latter recursion. Therefore, by weak duality, $\boldsymbol{\gamma}'\mathbf{d}$ and $\boldsymbol{\beta}'\mathbf{d}$ (where $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are the solutions to the above recursions) are lower bounds for $Z^*(\mathbf{d})$. Consequently, defining $\alpha_i = \min\{\gamma_i, \beta_i\}$, $\boldsymbol{\alpha}'\mathbf{d}$ is also a lower bound for $Z^*(\mathbf{d})$. □

We next derive the main structural results of this section. The first is a sufficient condition for the existence of a finite competitive ratio for an *arbitrary* online procurement strategy; this condition is subsequently applied to design a Make-to-Order strategy with a finite competitive ratio. The second structural result is a set of lower and upper bounds for the competitive ratio, if it exists. This result is subsequently applied to characterize the value of the competitive ratio of the aforementioned Make-to-Order strategy; when unit ordering cost and shortage parameters are identical over all periods, the lower and upper bounds coincide and we know the exact value of the competitive ratio for the Make-to-Order strategy. The following theorem is presented using the notation of Lemma 4.1.

Theorem 4.3. *For an arbitrary online strategy $\mathbf{q} \geq \mathbf{0}$, $\mathbf{b}'\mathbf{q} + \mathcal{K} \leq 0$ is a sufficient condition for the existence of a finite strict competitive ratio. Furthermore, if a strict finite ratio ρ exists, it satisfies*

$$\max\left\{\max_{1 \leq i \leq n}\left\{\frac{a_i}{c_i}\right\}, \frac{\mathbf{b}'\mathbf{q} + \mathcal{K}}{\mathbf{K}'\mathbf{e}}\right\} \leq \rho \leq \max_{1 \leq i \leq n}\left\{\frac{a_i}{\alpha_i}\right\}.$$

Proof. Utilizing the lower bound in Lemma 4.2, Lemma 4.1 and Assumption 2.3, an upper bound for the competitive ratio is

$$\rho \leq \sup_{\substack{\mathbf{d} \geq \mathbf{0} \\ \mathbf{d} \neq \mathbf{0}}} \left(\frac{\mathbf{a}'\mathbf{d} + \mathbf{b}'\mathbf{q} + \mathcal{K}}{\boldsymbol{\alpha}'\mathbf{d}}\right).$$

As the supremum in the upper bound is a linear-fractional program, it is equivalent to the following linear program

$$\begin{aligned} \max_{\mathbf{y},z} \quad & \mathbf{a}'\mathbf{y} + (\mathbf{b}'\mathbf{q} + \mathcal{K})z \\ \text{s.t.} \quad & \boldsymbol{\alpha}'\mathbf{y} = 1 \\ & \mathbf{y} \geq \mathbf{0}, z \geq 0, \end{aligned} \quad \text{whose dual is}$$

$$\begin{aligned} \min_{\beta} \quad & \beta \\ \text{s.t.} \quad & \beta\boldsymbol{\alpha} \geq \mathbf{a} \\ & 0 \geq \mathbf{b}'\mathbf{q} + \mathcal{K}. \end{aligned}$$

Technically, the vectors $\mathbf{a}$ and $\mathbf{b}$ are functions of the data $\mathbf{q}$ and variables $\mathbf{d}$ via the induced partition (P, N), so the optimization problem is actually *not* a linear program. However, a formal Lagrangian (weak) duality analysis arrives at the same conclusion and, for simplicity, we abuse notation and present the analysis as if $\mathbf{a}$ and $\mathbf{b}$ were constant vectors.

If the dual is feasible, a finite upper bound exists, which implies that a finite competitive ratio exists. The dual is feasible when $\mathbf{b}'\mathbf{q} + \mathcal{K} \leq 0$, which implies that the competitive ratio is at most the dual solution $\max_{1 \leq i \leq n}\{a_i/\alpha_i\}$. This proves the sufficiency of $\mathbf{b}'\mathbf{q} + \mathcal{K} \leq 0$ for a finite competitive ratio as well as the upper bound.

We next repeat the above analysis with the upper bound from Lemma 4.2 and obtain the following lower bound for the competitive ratio

$$\rho \geq \sup_{\mathbf{d} \geq \mathbf{0}} \left(\frac{\mathbf{a}'\mathbf{d} + \mathbf{b}'\mathbf{q} + \mathcal{K}}{\mathbf{c}'\mathbf{d} + \mathbf{K}'\mathbf{e}}\right),$$

which is equivalent to

$$\begin{aligned} \max_{\mathbf{y},z} \quad & \mathbf{a}'\mathbf{y} + (\mathbf{b}'\mathbf{q} + \mathcal{K})z \\ \text{s.t.} \quad & \mathbf{c}'\mathbf{y} + \mathbf{K}'\mathbf{e}z = 1 \\ & \mathbf{y} \geq \mathbf{0}, z \geq 0, \end{aligned} \quad \text{whose dual is}$$

$$\begin{aligned} \min_{\beta} \quad & \beta \\ \text{s.t.} \quad & \beta\mathbf{c} \geq \mathbf{a} \\ & \beta\mathbf{K}'\mathbf{e} \geq \mathbf{b}'\mathbf{q} + \mathcal{K}. \end{aligned}$$

Finally, the optimal dual solution

$$\beta = \max\left\{\max_{1 \leq i \leq n}\left\{\frac{a_i}{c_i}\right\}, \frac{\mathbf{b}'\mathbf{q} + \mathcal{K}}{\mathbf{K}'\mathbf{e}}\right\}$$

is a lower bound for the competitive ratio. □

4.1. Example: Make-to-Order policy

Next, we utilize the structural result in Theorem 4.3 to design a Make-to-Order procurement strategy; note that other strategies may be designed and this section simply serves as an illustration of how to apply the structural results. In particular, we base our design on the *sufficient* condition $\mathbf{b}'\mathbf{q} + \mathcal{K} \leq 0$, which leads naturally to a "backlog-up-to" policy. This strategy will avoid holding any inventory and will determine the best times to fulfill backlogged customer orders. For example, in Internet retailing, this model can determine the best time to fulfill orders received throughout the day or week.

Note that, since the strategy is online, the traditional offline constraint $I_n = 0$ is impossible to add to the strategy, since n is unknown. Therefore, it is possible that some demand at the end of the planning horizon will not be met. However, as can be seen in a single-period model, it is sometimes optimal to not serve all demand.

Theorem 4.4. *In period i,*

(1) *If $c_i \geq s_i$, order $q_i = 0$ units.*

(2) *If* $c_i < s_i$*:*
  (a) *If* $\sum_{j=1}^{i-1}(c_j - \sum_{k=j}^{i} s_k)q_j + (c_i - s_i)I_{i-1}^- + \sum_{j=1}^{i-1} K_j\delta(q_j) + K_i \le 0$, *order* $q_i = I_{i-1}^-$ *units.*
  (b) *Otherwise order* $q_i = 0$ *units.*

The strict competitive ratio ρ *of this strategy satisfies*

$$\max_{1\le i\le n}\left\{\frac{\sum_{j=i}^{n} s_j}{c_i}\right\} \le \rho \le \max_{1\le i\le n}\left\{\frac{\sum_{j=i}^{n} s_j}{\alpha_i}\right\}.$$

If there exist c *and* s *such that* $c_i = c$ *and* $s_i = s$ *for all* i*, then the strict competitive ratio*

$$\rho = \max\left\{\frac{sn}{c}, 1\right\}.$$

Proof. This strategy implies that $i \in N$ for all i; consequently, $a_i = \sum_{j=i}^{n} s_j$ and $b_i = c_i - \sum_{j=i}^{n} s_j$. If $c_i \ge s_i$, it is clearly cheaper to incur the inventory shortage cost in period i than to procure any product to satisfy demand; therefore, if $c_i \ge s_i$, it is optimal to procure zero units and wait until the cost structure is more economical (if ever).

We now consider the periods in which $c_i < s_i$. If ordering I_{i-1}^- takes place in period i, then the statement of the theorem indicates that

$$\begin{aligned} &\sum_{j=1}^{i-1}\left(c_j - \sum_{k=j}^{i} s_k\right) q_j + (c_i - s_i)I_{i-1}^- + \sum_{j=1}^{i-1} K_j\delta(q_j) + K_i \le 0 \\ &\iff \sum_{j=1}^{i}\left(c_j - \sum_{k=j}^{i} s_k\right) q_j + \sum_{j=1}^{i} K_j\delta(q_j) \le 0 \\ &\implies \sum_{j=1}^{i} b_j q_j + \sum_{j=1}^{i} K_j\delta(q_j) \le 0, \end{aligned} \tag{5}$$

where the last implication is due to $b_j = c_j - \sum_{k=j}^{n} s_k \le c_j - \sum_{k=j}^{i} s_k$. Let S index the periods where ordering takes place; mathematically, $S = \{i : q_i = I_{i-1}^-\}$ and let l denote the largest index in S (i.e., the last period where ordering occurs). Consequently,

$$\mathbf{b}'\mathbf{q} + \mathcal{K} = \sum_{j\in S}(b_j q_j + K_j) = \sum_{j=1}^{l}(b_j q_j + K_j\delta(q_j)) \le 0,$$

where the last inequality is due to Eq. (5) with $i = l$. Therefore, the invariant $\mathbf{b}'\mathbf{q} + \mathcal{K} \le 0$ is maintained. Theorem 4.3 implies that a finite competitive ratio exists and that the competitive ratio ρ satisfies

$$\max_{1\le i\le n}\left\{\frac{\sum_{j=i}^{n} s_j}{c_i}\right\} \le \rho \le \max_{1\le i\le n}\left\{\frac{\sum_{j=i}^{n} s_j}{\alpha_i}\right\}.$$

Finally, if there exist c and s such that $c_i = c$ and $s_i = s$ for all i, then the lower and upper bounds both equal

$$\max_{1\le i\le n}\left\{\frac{(n-i+1)s}{c}\right\} = \frac{sn}{c}. \quad \square$$

Corollary 4.5. *If all fixed ordering costs are zero* ($K_i = 0, \ \forall i$)*, then Statement 2. in Theorem 4.4 can be replaced with*

2. *If* $c_i < s_i$*, order* $q_i = I_{i-1}^-$ *units*

with no change in the remainder of the theorem.

Acknowledgements

We thank the anonymous referee, who corrected and suggested a valuable enhancement to Lemma 4.2, and whose other comments improved the clarity of the paper.

References

[1] S. Axsäter, Worst case performance for lot sizing heuristics, European Journal of Operational Research 9 (4) (1982) 339–343.
[2] S. Axsäter, Performance bounds for lot sizing heuristics, Management Science 31 (1985) 634–640.
[3] M. Ball, H. Gao, I. Karaesmen, Revenue management and competition: distribution-free methods, Technical report, University of Maryland, Robert H. Smith School of Business, 2008, working paper.
[4] M. Ball, M. Queyranne, Toward robust revenue management: competitive analysis of online booking, Research paper no. rhs 06-021, University of Maryland, Robert H. Smith School of Business, March 2006.
[5] G. Bitran, T. Magnanti, H. Yanasse, Approximation methods for the uncapacitated lot size problem, Management Science 30 (9) (1984) 1121–1140.
[6] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[7] N. Buchbinder, T. Kimbrelt, R. Levi, K. Makarychev, M. Sviridenko, Online Make-to-Order joint replenishment model: primal dual competitive algorithms, in: Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 2008 pp. 952–961.
[8] X. Chen, M. Sim, D. Simchi-Levi, P. Sun, Risk aversion in inventory management, Operations Research 55 (5) (2007) 828–842.
[9] L. Eeckhoudt, C. Gollier, H. Schlesinger, The risk-averse (and prudent) newsboy, Management Science 41 (5) (1995) 786–794.
[10] H. Gao, M. Ball, I. Karaesmen, Distribution-free methods for dynamic booking control, Technical report, University of Maryland, Robert H. Smith School of Business, 2008 (submitted for publication).
[11] G. Godfrey, W. Powell, An adaptive, distribution-free algorithm for the newsvendor problem with censored demands, with applications to inventory and distribution, Management Science 47 (8) (2001) 1101–1112.
[12] W.T. Huh, G. Janakiraman, J.A. Muckstadt, P. Rusmevichientong, An adaptive algorithm for finding the optimal base-stock policy in lost sales inventory systems with censored demand, Mathematics of Operations Research 34 (2) (2009) 397–416.
[13] W.T. Huh, R. Levi, P. Rusmevichientong, J.B. Orlin, Adaptive data-driven inventory control policies based on Kaplan-Meier estimator for censored demand, Operations Research (to appear).
[14] W.T. Huh, P. Rusmevichientong, A non-parametric asymptotic analysis of inventory planning with censored demand, Mathematics of Operations Research 34 (1) (2009) 103–123.
[15] Y. Lan, M. Ball, I. Karaesmen, Overbooking and fare-class allocation with limited information, Technical report, University of Maryland, Robert H. Smith School of Business, 2008 (submitted for publication).
[16] Y. Lan, H. Gao, M. Ball, I. Karaesmen, Revenue management with limited demand information, Management Science 54 (9) (2008) 1594–1609.
[17] H. Lau, The newsboy problem under alternative optimization objectives, Journal of the Operational Research Society 31 (1980) 525–535.
[18] R. Levi, R. Roundy, D. Shmoys, Provably near-optimal sampling-based policies for stochastic inventory control models, Mathematics of Operations Research 32 (4) (2007) 821–839.
[19] S. O'Neil, A. Chaudhary, Comparing online learning algorithms to stochastic approaches for the multi-period newsvendor problem, in: Proceedings of the Tenth Workshop on Algorithm Engineering and Experiments and the Fifth Workshop on Analytic Algorithmics and Combinatorics, 2008, SIAM pp. 49–63.
[20] R. Vachani, Performance of heuristics for the uncapacitated lot size problem, Naval Research Logistics 39 (1992) 801–813.
[21] W. van den Heuvel, A.P.M Wagelmans, Worst case analysis for a general class of on-line lot-sizing heuristics, Operations Research 58 (1) (2010) 59–67.
[22] J. Van Mieghem, Risk mitigation in newsvendor networks: resource diversification, flexibility, sharing, and hedging, Management Science 53 (8) (2007) 1269–1288.
[23] M.R. Wagner, Fully distribution-free profit maximization: the inventory management case, Mathematics of Operations Research 35 (4) (2010) 728–741.
[24] H. Wagner, T. Whitin, Dynamic version of the economic lot size model, Management Science 5 (1958) 89–96.
[25] W. Zangwill, A deterministic multi-period production scheduling model with backlogging, Management Science 13 (1) (1966) 105–119.