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Fully Distribution-Free Profit Maximization: The Inventory Management Case

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We study profit maximization in inventory control problems where demands are unknown. Neither probabilistic distributions nor sets are available to characterize the unknown demand parameters. Therefore, we adopt an online optimization perspective for our analysis. The usual competitive ratio is not well defined for the problems we analyze; consequently, we introduce a new worst-case metric that is suitable. We consider two inventory management frameworks: (1) perishable products with lost sales and (2) durable products with backlogged demand. We consider both finite and infinite planning horizons. We design best-possible online procurement strategies for all cases.

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1. Introduction. In this paper, we analyze various models of inventory control of a single product over finite or infinite planning horizons, where demands for the product are not known. In particular, in each period a decision maker must determine a procurement quantity without knowing the demand in that period. Relevant revenue and cost parameters are known a priori, and the objective is to maximize profit. Once a procurement decision is made in a given period, the demand in the period is revealed. However, there are neither probabilistic distributions nor sets available to characterize the demand uncertainty; therefore, stochastic and robust optimization techniques are not applicable. The problems we study are usually categorized as "online optimization" problems, and the methodology we use to study this type of problem is usually denoted "competitive analysis." In this paper, we design best-possible online procurement strategies for two inventory control problems: (1) perishable products with lost sales and (2) durable products with backlogged demand. We next survey relevant literature and then detail our contributions.

A decision-making strategy for an online problem is usually suboptimal, because not all information is available at the time a decision needs to be made. Therefore, the quality of online strategy is usually measured via the "competitive ratio," defined as the worst-case ratio of online to offline costs, where all data are available a priori in the offline solution. The idea to compare online and offline costs was originated by Sleator and Tarjan [28], and Karlin et al. [17] introduced the notion of the competitive ratio. For further details about online optimization and competitive analysis, see Borodin and El-Yaniv [6].

We assume that the reader has a basic knowledge of inventory management (for reference, see Axsäter [1]). Distribution-free approaches to inventory control usually make some assumptions about unknown demand. One popular approach is to model uncertain demand as a random variable and to assume that the mean and variance of the random variable are known. A min-max analysis, over all probabilistic distributions that have the given mean and variance, then usually follows. This approach was pioneered by Scarf [26], who derived an ordering rule for the newsvendor problem under this scenario. Gallego and Moon [10] investigated the recourse case, where a secondary procurement decision is allowed. Moon and Gallego [22] studied inventory management problems where the distribution of the lead time is unknown, but the mean and variance are given. Perakis and Roels [25] study the newsvendor problem where a demand distribution is unknown, but other information is available (as before, mean and variance, but also symmetry and/or unimodality of the distribution). Alternatively, Ma and Chen [21] assume lower and upper bounds on the demands, and their strategies and performance guarantees depend on these bounds. In revenue management, Lan et al. [19] also analyze the case where lower and upper demand bounds exist. Additionally, Ball and Queyranne [2] consider airfare booking when there is no distribution for the stream of customers purchasing airfare, and Gao et al. [11] consider distribution-free methods for booking control. Lan et al. [18] consider overbooking and fare-class allocation when there is limited demand information, and Ball et al. [3] consider distribution-free methods in revenue management and competition. Other examples of this style of research include Ehrhardt [9] and Natarajan et al. [23]. Buchbinder et al. [8] consider the online joint replenishment problem and provide constant competitive ratio results. Levi et al. [20] design a sampling-based procedure to overcome the absence of distributions, Godfrey and Powell [12] design an algorithm that directly estimates value functions without the need for demand distributions, and O'Neil and Chaudhary [24] compare online and stochastic algorithms for the multiperiod newsvendor problem. Huh and Rusmevichientong [13] and Huh et al. [16, 15] study various inventory management problems where a demand distribution is lacking but sales data are available; Huh and Rusmevichientong [14] apply these techniques to general stochastic optimization problems. Finally, Vairaktarakis [29], Bertsimas and Thiele [5], Ben-Tal et al. [4], and See and Sim [27] utilize robust optimization to study various problems in inventory management.

1.1. Contributions. The reader will see that, for the problems we study, the competitive ratio is not well defined, stemming from the fact that profits can be either positive or negative (i.e., a loss). Therefore, our first contribution is the introduction of a new metric of quality for online strategies that, to the best of our knowledge, has not appeared previously in the online optimization literature (although it has been utilized in the approximation algorithm literature). In particular, the worst-case measure we use is the maximum percentage of deviation between the online and optimal offline costs, which we denote as the "performance ratio." For the problems addressed in this paper, we analyze conditions for the existence of a finite performance ratio, and if it exists, we either derive its exact value or provide lower and upper bounds for it. We use duality theory to prove the majority of our results.

The first problem that we study consists of designing the procurement strategy for a single *perishable* product over a finite planning horizon. If demand exceeds product availability in any period, the excess demand is lost. We derive the procurement quantities in each period, dependent only on the period's unit revenue and fixed and unit ordering costs, that allow a finite performance ratio. We also derive the exact value of the performance ratio, which is dependent on only the unit revenues, the unit ordering, and the unit lost-sale costs. Furthermore, we show that this strategy is unique in the sense that *any* other strategy has an unbounded performance ratio.

The second problem that we study consists of designing the procurement and inventory management strategy for a single *durable* product that can be inventoried over either a finite or infinite planning horizon. Additionally, excess demand is backlogged for future periods. We prove that a finite performance ratio is impossible if either (1) fixed ordering costs are positive or (2) positive inventory is maintained. We then prove that, if fixed ordering costs are zero, a simple intuitive strategy has a finite performance ratio, which we bound from above and below. We also show that, despite not knowing the exact value of the performance ratio, there does not exist another strategy with a strictly smaller performance ratio. In other words, our strategy is the best possible. We consider the infinite horizon case for a special cost structure, and we prove comparable results.

Finally, we contrast my paper with Wagner [30], which analyzes the models in this paper when revenues are absent. Since the objective is to minimize total inventory management *cost*, the competitive ratio is applicable and used in Wagner [30], whereas it is not in this paper. Interestingly, even for single-period models, the best-possible procurement strategies are very different; e.g., for maximizing profit, as detailed above, we are given a unique ordering quantity whereas, for minimizing cost, Wagner [30] proves that there exists an *interval* of valid ordering quantities.

Outline of this paper. In §2 we explain the notation that we used in the paper, derive the two models we study, discuss in detail the difference between online and offline problems and, finally, introduce the performance ratio that is my metric of quality. In §3, we analyze the perishable product with lost sales model, first for a single period and then for an arbitrary number of periods. Finally, in §4, we analyze the durable product with backlogged demand model, first for a finite planning horizon and then for an infinite horizon.

2. Preliminaries

2.1. Notation. We begin by explaining the notation that we use throughout the paper. Scalar values are represented in regular type, and vectors are represented in boldface type. For example, $\mathbf{x} = (x_1, \ldots, x_n)$ is a column vector of n elements, whereas $\mathbf{x}'$ is the transpose (row) vector of the (column) vector $\mathbf{x}$. Additionally, we define $\mathbf{e} = (1, \ldots, 1)$ as the n-dimensional vector of all ones. Finally, $\delta(x)$ is the indicator function; i.e., $\delta(x) = 1$ if $x > 0$, and $\delta(x) = 0$ if $x = 0$.

We next define notational shortcuts that streamline many proofs for an n-dimensional vector (of variables or parameters) $\mathbf{x} = (x_1, \ldots, x_n)$:

(i) Let $\mathbf{x}^{+1}$ be defined as $x_i^{+1} = x_{i+1}$ for $i \le n-1$ and $x_n^{+1} = 0$;
(ii) Let $\mathbf{x}^{-1}$ be defined as $x_i^{-1} = x_{i-1}$ for $i \ge 2$ and $x_1^{-1} = 0$;
(iii) Let $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_n)$ where $\hat{x}_i = \sum_{j=1}^{i} x_j$;

(iv) Let $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ where $\tilde{x}_i = \sum_{j=i}^n x_j$;

(v) Let $\mathbf{x}^+ = \max\{\mathbf{x}, \mathbf{0}\}$ and $\mathbf{x}^- = \max\{-\mathbf{x}, \mathbf{0}\}$, where the max and min operators are defined componentwise.

Definitions 3 and 4 lead to a useful identity, which we use frequently in proofs. For any n-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{x}'\hat{\mathbf{y}} = \tilde{\mathbf{x}}'\mathbf{y}, \tag{1}$$

which is easily justified by inverting the order of the summations: $\mathbf{x}'\hat{\mathbf{y}} = \sum_{i=1}^n x_i \sum_{j=1}^i y_j = \sum_{i=1}^n y_i \sum_{j=i}^n x_j = \mathbf{y}'\tilde{\mathbf{x}}$. Next, we provide a useful result that links a specific linear-fractional program with linear programming.

Lemma 2.1 (Linear Fractional Programming). *If $\{\mathbf{x}\colon \mathbf{f}'\mathbf{x} + g > 0,\ \mathbf{x} \geq \mathbf{0}\}$ is nonempty, then the optimization problems*

$$\begin{aligned} \max_x \quad & \frac{\mathbf{c}'\mathbf{x} + d}{\mathbf{f}'\mathbf{x} + g} \\ \text{s.t.} \quad & \mathbf{f}'\mathbf{x} + g > 0 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \max_{y,\,z} \quad & \mathbf{c}'\mathbf{y} + dz \\ \text{s.t.} \quad & \mathbf{f}'\mathbf{y} + gz = 1 \\ & \mathbf{y} \geq \mathbf{0},\ z \geq 0 \end{aligned}$$

are equivalent.

Proof of Lemma 2.1. A proof of a more general result can be found in Boyd and Vandenberghe [7, p. 51]. □

2.2. Model derivation. We begin by detailing the data for the problems. We consider the n-period inventory management of a single product where the objective is to maximize total profit. In period i, $d_i \geq 0$ is the demand for the product, $q_i \geq 0$ is the ordering quantity, $c_i \geq 0$ is the unit ordering cost, $K_i \geq 0$ is the fixed ordering cost for placing an order, $r_i \geq 0$ is the unit revenue, $h_i \geq 0$ is the unit inventory holding cost, and $s_i \geq 0$ is the unit inventory shortage cost. In vector notation, the parameters are $\mathbf{d}, \mathbf{q}, \mathbf{c}, \mathbf{K}, \mathbf{r}, \mathbf{h}, \mathbf{s} \geq \mathbf{0}$. We next make two assumptions that hold for the sequel. We first assume that the revenues and variable costs are positive (we make no assumption regarding the fixed ordering costs).

Assumption 2.1. $\mathbf{r}, \mathbf{c}, \mathbf{h}, \mathbf{s} > \mathbf{0}$.

We also assume that the margin in any period is positive.

Assumption 2.2. $\mathbf{r} - \mathbf{c} > \mathbf{0}$.

We next derive the models for two distinctly different operational environments.

2.2.1. Perishable products with lost sales. We begin with the case where the product is perishable (e.g., produce products) and can not be inventoried for future periods. Additionally, we assume that any unmet demand is lost forever. The inventory holding costs $h_i \geq 0$ have the managerial interpretation of a write-off (of perished inventory) cost, and the inventory shortage costs $s_i \geq 0$ have the interpretation of quantifying lost sales. In period i, the revenue generated is $r_i \min\{q_i, d_i\}$, the write-off cost is $h_i(q_i - d_i)^+$, the shortage cost is $s_i(d_i - q_i)^+$, and the ordering costs are $c_i q_i + K_i \delta(q_i)$. The model in this case is

$$\max_{q \geq 0} \sum_{i=1}^n (r_i \min\{q_i, d_i\} - h_i(q_i - d_i)^+ - s_i(d_i - q_i)^+ - c_i q_i - K_i \delta(q_i)). \tag{2}$$

2.2.2. Durable products with backlogging. The second scenario we study is one where the product is not perishable and can be inventoried to satisfy demand in later periods. Additionally, if demand exceeds available inventory, it is backlogged and satisfied in future periods.

We define the inventory I_i *at the end* of period i, which must satisfy the inventory balance constraints $I_i = I_i + q_i - d_i$, for $i = 1, \ldots, n$, with initial inventory $I_0 = 0$. We next explicitly define positive and negative inventory as $I_i^+ = \max\{I_i, 0\}$ and $I_i^- = \max\{-I_i, 0\}$, respectively. Clearly, $I_i = I_i^+ - I_i^-$ and $|I_i| = I_i^+ + I_i^-$. Finally, note that we can substitute out the expression for I_i by using the identity $I_i = \sum_{j=1}^i (q_j - d_j) = \hat{q}_i - \hat{d}_i$; this identity is used in subsequent proofs.

In period i, the revenue generated is $r_i \min\{q_i + I_{i-1}^+, d_i + I_{i-1}^i\}$. To see this, note that the inventory available in period i is $q_i + I_{i-1}^+$ and the total demand (current and backlogged) is $d_i + I_{i-1}^-$. In period i, the inventory holding cost is $h_i I_i^+$, the inventory shortage cost is $s_i I_i^-$, and the inventory ordering cost is $K_i\delta(q_i)+c_i q_i$, where δ is the indicator function. In summary, the model in this case is

$$\begin{aligned} \max\ & \sum_{i=1}^{n}(r_i \min\{q_i+I_{i-1}^+, d_i+I_{i-1}^-\} - h_i I_i^+ - s_i I_i^- - c_i q_i - K_i\delta(q_i)),\\ \text{s.t. } & I_i = I_{i-1}+q_i-d_i, \quad i=1,\dots,n, \quad I_0=0, \\ & q_i \ge 0, \quad i=1,\dots,n. \end{aligned} \tag{3}$$

2.3. Online and offline problems. In this paper we design strategies to make decisions without knowing customer demands. In particular, in period i, an *online player* must decide how much product to order q_i without knowing the demand d_i. However, this online player does know the cost and revenue structure of all periods $j \le i$, as well as all previous demands d_j and previous decisions q_j for $j < i$. Additionally, the online player does not know how many periods n there are in the planning horizon. The profit that an online player accrues over n periods will be denoted as $Z(\mathbf{d})$. An *offline adversary* knows all demands $\mathbf{d}$ a priori and makes optimal decisions. The profit that an offline adversary accrues over n periods will be denoted as $Z^*(\mathbf{d})$. In particular, $Z^*(\mathbf{d})$ is defined as the optimal solution of either Model (2) or Model (3) (the relevant model will be clear from context). Clearly $Z^*(\mathbf{d}) \ge Z(\mathbf{d})$. It will also be convenient to write $Z(\mathbf{d}) = \sum_{i=1}^n Z_i(d_i)$, where $Z_i(d_i)$ is the profit contribution in period i; similarly, $Z^*(\mathbf{d}) = \sum_{i=1}^n Z_i^*(d_i)$.

Traditionally, the quality of an online strategy was measured via the *competitive ratio*. In our notation and with the additional requirement that $Z(\mathbf{d}) \ge 0$ for any $\mathbf{d} \ge \mathbf{0}$ (e.g., for a *revenue* maximization problem), the competitive ratio would be defined as the largest value of $\alpha \in [0, 1]$ such that

$$Z(\mathbf{d}) \ge \alpha Z^*(\mathbf{d}), \ \ \forall\, \mathbf{d} \ge \mathbf{0} \quad \text{or equivalently} \quad \alpha = \inf_{\substack{Z^*(d)>0\\ d\ge 0}} \frac{Z(\mathbf{d})}{Z^*(\mathbf{d})};$$

note that it is without loss of generality to assume $Z^*(\mathbf{d}) > 0$ in the latter definition because, for $\mathbf{d}$ such that $Z^*(\mathbf{d}) = 0$, we also have that $Z(\mathbf{d}) = 0$, which implies $\alpha = 1$ for these $\mathbf{d}$. In other words, if an online revenue maximization strategy had a competitive ratio α, we could guarantee at least a fraction α of the maximum revenue possible, for *any* demand realization $\mathbf{d}$.

However, in this paper we allow for the possibility that the offline adversary makes positive profit $Z^*(\mathbf{d}) > 0$ and the online player experiences a loss $Z(\mathbf{d}) < 0$ for some $\mathbf{d}$. Indeed, if an online algorithm ever orders a positive quantity, there always exists a demand sequence that forces the online player to accept a loss, while the offline adversary retains neutral or positive profit. Therefore, the competitive ratio is not well defined for these models.

Instead, we use a worst-case measure that is defined as the maximum percent deviation from the optimal solution:

$$\rho \triangleq \sup_{d\ge 0} \frac{Z^*(\mathbf{d}) - Z(\mathbf{d})}{Z^*(\mathbf{d})} = 1 + \sup_{d\ge 0} \frac{-Z(\mathbf{d})}{Z^*(\mathbf{d})}. \tag{4}$$

We now make two additional assumptions, the first to avoid trivialities and the second to avoid singularities. We first assume that there is nonzero demand in at least one period.

Assumption 2.3. $\mathbf{d} \ne \mathbf{0}$.

We next assume that it is possible to make positive profit over the entire planning horizon.

Assumption 2.4. $Z^*(\mathbf{d}) > 0$, $\forall\, \mathbf{d} \ge \mathbf{0}$, $\mathbf{d} \ne \mathbf{0}$.

The latter assumption implies that $\rho \ge 0$. We denote ρ as the *performance ratio* of an online profit maximization strategy; clearly, the smaller ρ is, the better. Finally, if an online strategy has performance ratio ρ and we can show that no other strategy has a strictly smaller performance ratio, then we say that the original strategy is *best-possible*.

Next, we return for a moment to the competitive ratio and note that it is defined more generally as the largest value of $\alpha \in [0, 1]$ such that $Z(\mathbf{d}) \ge \alpha Z^*(\mathbf{d}) + \beta$, where β is some constant independent of $\mathbf{d}$. If $\beta \le 0$, we say that c is a *strict* competitive ratio. We apply a similar approach for the performance ratio. Note that

$$\begin{aligned} \rho = \sup_{d\ge 0} \frac{Z^*(\mathbf{d}) - Z(\mathbf{d})}{Z^*(\mathbf{d})} \iff\ & \rho \text{ is minimal value such that } \frac{Z^*(\mathbf{d}) - Z(\mathbf{d})}{Z^*(\mathbf{d})} \le \rho, \ \ \forall\, \mathbf{d} \ge \mathbf{0}, \\ \iff\ & \rho \text{ is minimal value such that } -Z(\mathbf{d}) \le (\rho - 1) Z^*(\mathbf{d}), \ \ \forall\, \mathbf{d} \ge \mathbf{0}. \end{aligned}$$

Therefore, we generalize the definition of the performance ratio to the smallest value of $\rho \geq 0$ such that

$$-Z(\mathbf{d}) \leq (\rho - 1)Z^*(\mathbf{d}) + \beta, \quad \forall\, \mathbf{d} \geq \mathbf{0}, \tag{5}$$

where β is a constant independent of $\mathbf{d}$; if $\beta \leq 0$, we say that the performance ratio ρ is *strict*. We also say that ρ is an *asymptotic* performance ratio if there exists n_0 such that Definition (5) is valid for all problems with planning horizons $n \geq n_0$.

Finally, a common interpretation of online analysis is via game theory. To see this, we expand the powers of the offline adversary: not only does the offline adversary know the demands a priori, but also he/she can in fact *choose* the demands. The two players now have conflicting goals: The online player wants to *minimize* ρ by choosing the online ordering quantities q intelligently, and the offline adversary wants to *maximize* ρ by choosing the demands d and the offline ordering quantities appropriately.

3. Perishable products with lost sales.

In this section, we study the inventory management problem where the product is perishable and unmet demands are lost forever, and we derive the *unique* procurement strategy that has a finite performance ratio. We note that Model (2) is separable and can be written as

$$\sum_{i=1}^{n} \max_{q_i \geq 0} (r_i \min\{q_i, d_i\} - h_i(q_i - d_i)^+ - s_i(d_i - q_i)^+ - c_i q_i - K_i \delta(q_i));$$

therefore, we begin by studying the single period case.

3.1. Single period analysis.

For a single period, and omitting subscripts, Model (2) is written as

$$\max_{q \geq 0} (r \min\{q, d\} - h(q - d)^+ - s(d - q)^+ - cq - K\delta(q))$$

and is similar to the familiar newsvendor problem. The next result derives the only ordering quantity that induces a finite performance ratio.

THEOREM 3.1. *The ordering quantity $q = K/(r - c)$ has a strict performance ratio equal to*

$$1 + \frac{s}{r - c}.$$

Furthermore, all other ordering quantities have an unbounded performance ratio.

PROOF OF THEOREM 3.1. As a function of d, the optimal offline adversarial solution is to either order $q = d$ or $q = 0$, which implies $Z^*(d) = \max\{(r - c)d - K, -sd\}$; Assumption 2.4 implies that $Z^* = (r - c)d - K > 0$, which also gives us the strict lower bound $d > K/(r - c)$. The performance ratio in this case is therefore

$$\rho = 1 + \sup_{d > K/(r-c)} \frac{-Z(d)}{Z^*(d)} = 1 + \sup_{d > K/(r-c)} \frac{-Z(d)}{(r - c)d - K}.$$

We consider two possible cases that determine the structure of $-Z(d)$, where q is now the online player's procurement quantity. We assume that $q > 0$ and consider $q = 0$ subsequently.

Case 1 ($q > d$): We have that

$$\rho - 1 = \sup_{d > K/(r-c)} \frac{-(r + h)d + (c + h)q + K}{(r - c)d - K},$$

the right-hand-side of which is a linear-fractional program which, applying Lemma 2.1, can be written as the linear program

$$\left\{ \begin{array}{ll} \max\limits_{y,z} & -(r + h)y + ((c + h)q + K)z \\ \text{s.t.} & (r - c)y - Kz = 1 \\ & y, z \geq 0 \end{array} \right\} \quad \text{whose dual is} \quad \left\{ \begin{array}{ll} \min\limits_{\alpha} & \alpha \\ \text{s.t.} & \alpha(r - c) \geq -(r + h) \\ & -\alpha K \geq (c + h)q + K \end{array} \right\}.$$

The primal linear program is clearly feasible; for a finite performance ratio, the dual must also be feasible, which implies

$$-\frac{r + h}{r - c} \leq -\frac{c + h}{K} q - 1 \iff q \leq \frac{K}{r - c}. \tag{6}$$

Case 2 ($q \le d$): We have that

$$\rho - 1 = \sup_{d > K/(r-c)} \frac{-(r-c+s)q + sd + K}{(r-c)d - K},$$

the right-hand-side of which is a linear-fractional program, which can be written as the linear program

$$\left\{\begin{array}{ll} \max\limits_{y,z} & sy + (K - (r-c+s)q)z \\ \text{s.t.} & (r-c)y - Kz = 1 \\ & y, z \ge 0 \end{array}\right\} \quad \text{whose dual is} \quad \left\{\begin{array}{ll} \min\limits_{\alpha} & \alpha \\ \text{s.t.} & \alpha(r-c) \ge s \\ & -\alpha K \ge K - (r-c+s)q \end{array}\right\}.$$

As before, the primal linear program is clearly feasible; for a finite performance ratio, the dual must also be feasible, which implies that

$$\frac{s}{r-c} \le \frac{r-c+s}{K} q - 1 \iff q \ge \frac{K}{r-c}. \tag{7}$$

Finally, if $q = 0$, we set $K - (r-c+s)q = 0$ and the dual in Case 2 ($q < d$) simplifies to $\min\{\alpha: \alpha \ge s/(r-c), \alpha \le 0\}$, which is not feasible and, hence, no finite performance ratio exists.

Since a priori an online player does not know if Case 1 or Case 2 will occur, the conditions (6) and (7) in both cases must be satisfied; therefore, to maintain a finite performance ratio, $q = K/(r-c)$. Finally, the performance ratio can be calculated as one plus the maximum of the dual solutions:

$$1 + \max\left\{-\frac{r+h}{r-c}, \frac{s}{r-c}\right\} = 1 + \frac{s}{r-c}. \quad \square$$

3.2. Multiple period analysis. We next show that for multiple periods, the online player must order $q_i = K_i/(r_i - c_i)$ in each period i; otherwise, no finite performance ratio is possible. At first glance, this result might seem an obvious extension of the single period result in Theorem 3.1. However, Assumption 2.4 implies that $d > K/(r-c)$ when there is a single period and we *do not* assume $d_i > K_i/(r_i - c_i)$ in each period i. The main result of this section is the following theorem, which is proved via two subsequent lemmas.

Theorem 3.2. *The online strategy of ordering $q_i = K_i/(r_i - c_i)$ in period i, for $i = 1, \ldots, n$, has a performance ratio equal to*

$$1 + \max_{1 \le i \le n} \frac{s_i}{r_i - c_i}.$$

Furthermore, any other online procurement strategy has an unbounded performance ratio.

Lemma 3.1. *The online player must order $q_i = K_i/(r_i - c_i)$ in period i, for $i = 1, \ldots, n$, to maintain a finite performance ratio. If there is one period in which the ordering quantities differ, the performance ratio is unbounded.*

Proof of Lemma 3.1. Consider an online strategy $\mathbf{q}$ where j is the first period where $q_j \ne K_j/(r_j - c_j)$. The offline adversary sets $d_i = 0$ for $i \ne j$, and we consider the behavior of the performance ratio as $d_j \to K_j/(r_j - c_j)$ from above; Assumption 2.4 is clearly not violated by these conditions. For convenience, we define

$$\Omega_j \triangleq \{\mathbf{d} \ge \mathbf{0}: d_i = 0, i \ne j, d_j > K_j/(r_j - c_j)\}.$$

Recall that $Z(\mathbf{d}) = \sum_{i=1}^n Z_i(d_i)$ and $Z^*(\mathbf{d}) = \sum_{i=1}^n Z_i^*(d_i)$. The online profits in periods $i \ne j$ are clearly nonpositive; therefore,

$$Z(\mathbf{d}) \le r_j \min\{q_j, d_j\} - h_j(q_j - d_j)^+ - s_j(d_j - q_j)^+ - c_j q_j - K_j \delta(q_j) = Z_j(d_j).$$

Letting $\check{q}_i$, $\forall i$ denote the optimal offline adversarial procurement quantities, we notice that, for $\mathbf{d} \in \Omega_j$, $\check{q}_i = 0$ for $i \ne j$ and

$$Z^*(\mathbf{d}) = \max_{\check{q}_j \ge 0} (r_j \min\{\check{q}_j, d_j\} - h_j(\check{q}_j - d_j)^+ - s_j(d_j - \check{q}_j)^+ - c_j \check{q}_j - K_j \delta(\check{q}_j)) = Z_j^*(d_j).$$

We next give a lower bound for the performance ratio as follows:

$$\rho = 1 + \sup_{\substack{\mathbf{d} \ge 0 \\ \mathbf{d} \ne 0}} \frac{-Z(\mathbf{d})}{Z^*(\mathbf{d})} \ge 1 + \sup_{\mathbf{d} \in \Omega_j} \frac{-Z(\mathbf{d})}{Z^*(\mathbf{d})} \ge 1 + \sup_{\mathbf{d} \in \Omega_j} \frac{-Z_j(d_j)}{Z^*(\mathbf{d})} = 1 + \sup_{d_j > K_j/(r_j - c_j)} \frac{-Z_j(d_j)}{Z_j^*(d_j)}.$$

Because $q_j \neq K_j/(r_j - c_j)$, we know, via Theorem 3.1, that

$$\sup_{d_j > K_j/(r_j - c_j)} \frac{-Z_j(d_j)}{Z_j^*(d_j)}$$

is unbounded, and the proof is complete. □

Lemma 3.2. *The performance ratio ρ of ordering $q_i = K_i/(r_i - c_i)$ in each period i is equal to*

$$1 + \max_{1 \le i \le n} \frac{s_i}{r_i - c_i}.$$

Proof of Lemma 3.2. We begin by recalling that $Z(\mathbf{d}) = \sum_{i=1}^n Z_i(d_i)$ and $Z^*(\mathbf{d}) = \sum_{i=1}^n Z_i^*(d_i)$. Let $S = \{i\colon d_i > K_i/(r_i - c_i)\}$ denote the set of periods where positive profit is possible; Assumption 2.4 implies that S is nonempty. Clearly, for $i \in S$, $Z_i^*(d_i) = (r_i - c_i)d_i - K_i$, and for $i \notin S$, $Z_i^*(d_i) = 0$. Additionally, for $i \in S$, Theorem 3.1 shows that $-Z_i(d_i) \le (s_i/(r_i - c_i))Z^*(d_i)$ for all $d_i > K_i/(r_i - c_i)$.

We first prove an upper bound on ρ. For $i \notin S$, $d_i \le K_i/(r_i - c_i) = q_i$, the online ordering quantities. Therefore, for $i \notin S$,

$$Z_i(d_i) = r_i d_i - c_i \frac{K_i}{r_i - c_i} - h_i\left(\frac{K_i}{r_i - c_i} - d_i\right) - K_i = (r_i + h_i)d_i - (c_i + h_i)\frac{K_i}{r_i - c_i} - K_i.$$

We can therefore find a demand-independent upper bound on the negation

$$-Z_i(d_i) \le (c_i + h_i)\frac{K_i}{r_i - c_i} + K_i \triangleq \gamma_i,$$

for $i \notin S$. Consequently, defining the demand-independent constant $\gamma = \sum_{i \notin S} \gamma_i$, we have that

$$\begin{aligned} -Z(\mathbf{d}) &= -\sum_{i \in S} Z_i(d_i) - \sum_{i \notin S} Z_i(d_i) \\ &\le \sum_{i \in S}\left(\frac{s_i}{r_i - c_i}\right) Z_i^*(d_i) + \gamma \\ &\le \max_{j \in S}\left\{\frac{s_j}{r_j - c_j}\right\} \sum_{i \in S} Z_i^*(d_i) + \gamma \\ &= \max_{i \in S}\left\{\frac{s_i}{r_i - c_i}\right\} Z^*(\mathbf{d}) + \gamma \\ &\le \max_{1 \le i \le n}\left\{\frac{s_i}{r_i - c_i}\right\} Z^*(\mathbf{d}) + \gamma; \end{aligned}$$

therefore, referring to Definition 5, we see that $\rho \le 1 + \max_{1 \le i \le n}\{s_i/(r_i - c_i)\}$. To show tightness of this bound, we pick an arbitrary $i \in \{1, \ldots, n\}$, we let $d_j = 0$ for $j \neq i$ and pick some value of $d_0 > K_i/(r_i - c_i)$. For $\lambda \ge 0$, $d_i = d_0 + \lambda/(r_i - c_i) > K_i/(r_i - c_i)$ and, consequently, $Z^*(\mathbf{d}) = Z_i^*(d_i) = (r_i - c_i)d_i - K_i$ for all $\mathbf{d}$ satisfying these conditions. For $j \neq i$, the online profit in period j satisfies

$$-Z_j(d_j) = (c_j + h_j)\frac{K_j}{r_j - c_j} + K_j,$$

and in period i,

$$-Z_i(d_i) = s_i d_i - (r_i - c_i + s_i)\frac{K_i}{r_i - c_i} + K_i.$$

As $\lambda \to \infty$, the ratio of $-Z(\mathbf{d}) = -\sum_{i=1}^n Z_i(d_i)$ to $Z^*(\mathbf{d}) = (r_i - c_i)d_i - K_i$ approaches $s_i/(r_i - c_i)$ arbitrarily closely. Because i was chosen arbitrarily, we can get the ratio $-Z(\mathbf{d})/Z^*(\mathbf{d})$ arbitrarily close to $\max_{1 \le i \le n}\{s_i/(r_i - c_i)\}$, which implies that the performance ratio is at least $1 + \max_{1 \le i \le n}\{s_i/(r_i - c_i)\}$. □

4. Durable products with backlogging. In this section, we study Model (3)

$$\begin{aligned} Z^* = \max \ & \sum_{i=1}^{n} (r_i \min\{q_i + I_{i-1}^+, d_i + I_{i-1}^-\} - h_i I_i^+ - s_i I_i^- - c_i q_i - K_i \delta(q_i)), \\ \text{s.t.} \ & I_i = I_{i-1} + q_i - d_i, \quad i = 1, \ldots, n, \\ & q_i \geq 0, \quad i = 1, \ldots, n. \end{aligned}$$

in detail. We prove a number of results, many of which are negative. We begin by considering in §4.1 the case where the fixed ordering costs K_i are zero. We show that the online cost of an arbitrary online procurement strategy $\mathbf{q}$ is a linear combination of $\mathbf{q}$ and the demands $\mathbf{d}$. We then give a necessary and sufficient condition for the arbitrary online strategy $\mathbf{q}$ to have a finite performance ratio, and we characterize the ratio. Next, in §4.2, we show that if fixed ordering costs are positive ($K_i > 0$, $\forall i$), no finite performance ratio can be guaranteed. Next, in §4.3, we show that if positive inventory is maintained in any period, no finite performance ratio can be guaranteed. We then give an intuitive online strategy, whose finite performance ratio we bound from above and below. Finally, we consider a special cost structure and study the asymptotic case in §4.4.

Throughout the analysis of model (3), the following notions are important and useful. Let $P = \{i\colon I_i \geq 0\}$ and $N = \{i\colon I_i \leq 0\}$ denote the periods of nonnegative and nonpositive inventory, respectively. If inventory is zero in period i, we can assign i arbitrarily to P or N. Clearly, P and N are subsets of $\{1, \ldots, n\}$, so if $i \notin \{1, \ldots, n\}$, then $i \notin N \vee P$. Finally, we will also require the following lemma that provides lower and upper bounds on the optimal offline adversarial cost; the proof is provided in the appendix.

Lemma 4.1. *The optimal offline cost of model* (3) *has the following upper bound*

$$Z^*(\mathbf{d}) \leq (\mathbf{r} - \mathbf{c} + \omega \mathbf{e})'\mathbf{d},$$

where $\omega = \sum_{i=1}^{n} \max\{0, -(r_i - r_{i+1}) + (c_i - c_{i+1}) - s_i, r_i - (c_i - c_{i+1}) - h_i\}$. *Furthermore, if* $K_i = 0$ *for all* i, *the optimal offline cost of model* (3) *has the following lower bound*:

$$Z^*(\mathbf{d}) \geq (\mathbf{r} - \mathbf{c})'\mathbf{d}.$$

4.1. General analysis for zero fixed ordering costs. In this section we assume that fixed ordering costs are zero; i.e., $K_i = 0$ for all i. Letting $\mathbf{1}_{\{i \in S\}}$ denote the indicator function for whether $i \in S$, we have the following linear combination characterization of the cost of an arbitrary online strategy. Note that a summation over an empty set is defined as zero.

Lemma 4.2. *For an arbitrary online strategy* $\mathbf{q} \geq \mathbf{0}$, *we can write the online cost as*

$$Z(\mathbf{d}) = \mathbf{a}'\mathbf{d} + \mathbf{b}'\mathbf{q},$$

where

$$a_i = r_i \mathbf{1}_{\{i \in P\}} + \sum_{\substack{j=i \\ j \in P}}^{n} h_j + \sum_{\substack{j=i-1 \\ j \in P \\ j-1 \in N}}^{n} r_j - \sum_{\substack{j=i \\ j \in N}}^{n} s_j - \sum_{\substack{j=i-1 \\ j \in N \\ j-1 \in P}}^{n} r_j$$

and

$$b_i = r_i \mathbf{1}_{\{i \in N\}} - c_i + \sum_{\substack{j=i \\ j \in N}}^{n} s_j + \sum_{\substack{j=i-1 \\ j \in N \\ j-1 \in P}}^{n} r_j - \sum_{\substack{j=i \\ j \in P}}^{n} h_j - \sum_{\substack{j=i-1 \\ j \in P \\ j-1 \in N}}^{n} r_j,$$

for $i = 1, \ldots, n$.

Proof of Lemma 4.2. We begin by noting that the periods where backlogged demand drives the revenue generation correspond exactly to those periods that end with nonnegative inventory:

$$\{i\colon q_i + I_{i-1}^+ \geq d_i + I_{i-1}^-\} = \{i\colon I_{i-1}^+ - I_{i-1}^- + q_i - d_i \geq 0\} = P.$$

Decomposing along the partition (P, N), we have that the online cost

$$Z(\mathbf{d}) = \sum_{i \in P} (r_i(d_i + I_{i-1}^-) - h_i I_i) + \sum_{i \in N} (r_i(q_i + I_{i-1}^+) + s_i I_i) - \mathbf{c}'\mathbf{q}, \tag{8}$$

since $I_i^+ = I_i$ for $i \in P$ and $I_i^- = -I_i$ for $i \in N$. We first consider $i \in P$ and see that the cost contribution from periods ending with nonnegative inventory can be written as

$$\begin{aligned}\sum_{i\in P}(r_i(d_i + I_{i-1}^-) - h_i I_i) &= \sum_{i\in P}(r_i d_i - h_i I_i) - \sum_{\substack{i\in P\\ i-1\in N}} r_i I_{i-1}\\ &= \sum_{i\in P}(r_i d_i - h_i(\hat{q}_i - \hat{d}_i)) - \sum_{\substack{i\in P\\ i-1\in N}} r_i(\hat{q}_{i-1} - \hat{d}_{i-1}),\end{aligned} \tag{9}$$

where the second equality is obtained via the identities $I_i = \sum_{j=1}^{i}(q_j - d_j) = \hat{q}_i - \hat{d}_i$. Expanding and inverting summations (e.g., $\sum_{i\in P} h_i \hat{q}_i = \sum_{i\in P} h_i \sum_{j=1}^{i} q_j = \sum_{i=1}^{n} q_i \sum_{j=i,\, j\in P}^{n} h_j$), we have that Expression (9) is equal to

$$\begin{aligned}&\sum_{i\in P} r_i d_i - \sum_{i=1}^{n} q_i \sum_{\substack{j=i\\ j\in P}}^{n} h_j + \sum_{i=1}^{n} d_i \sum_{\substack{j=i\\ j\in P}}^{n} h_j - \sum_{i=1}^{n} q_i \sum_{\substack{j=i-1\\ j\in P\\ j-1\in N}}^{n} r_j + \sum_{i=1}^{n} d_i \sum_{\substack{j=i-1\\ j\in P\\ j-1\in N}}^{n} r_j\\ &= \sum_{i\in P} r_i d_i - \mathbf{q}'\mathbf{f} + \mathbf{d}'\mathbf{f} - \mathbf{q}'\mathbf{g} + \mathbf{d}'\mathbf{g},\end{aligned} \tag{10}$$

where $f_i = \sum_{j=i,\, j\in P}^{n} h_j$ and $g_i = \sum_{j=i-1,\, j\in P,\, j-1\in N}^{n} r_j$. Similarly, for $i \in N$, the cost contribution from periods ending with nonpositive inventory can be written as

$$\begin{aligned}\sum_{i\in N}(r_i(q_i + I_{i-1}^+) + s_i I_i) &= \sum_{i\in N}(r_i q_i + s_i I_i) + \sum_{\substack{i\in N\\ i-1\in P}} r_i I_{i-1}\\ &= \sum_{i\in N}(r_i q_i + s_i(\hat{q}_i - \hat{d}_i)) + \sum_{\substack{i\in N\\ i-1\in P}} r_i(\hat{q}_{i-1} - \hat{d}_{i-1})\\ &= \sum_{i\in N} r_i q_i + \sum_{i=1}^{n} q_i \sum_{\substack{j=i\\ j\in N}}^{n} s_j - \sum_{i=1}^{n} d_i \sum_{\substack{j=i\\ j\in N}}^{n} s_j + \sum_{i=1}^{n} q_i \sum_{\substack{j=i-1\\ j\in N\\ j-1\in P}}^{n} r_j - \sum_{i=1}^{n} d_i \sum_{\substack{j=i-1\\ j\in N\\ j-1\in P}}^{n} r_j\\ &= \sum_{i\in N} r_i q_i + \mathbf{q}'\mathbf{u} - \mathbf{d}'\mathbf{u} + \mathbf{q}'\mathbf{v} - \mathbf{d}'\mathbf{v},\end{aligned} \tag{11}$$

where $u_i = \sum_{j=i,\, j\in N}^{n} s_j$ and $v_i = \sum_{j=i-1,\, j\in N,\, j-1\in P}^{n} r_j$. Consequently, using Equations (8), (10), and (11), we obtain that

$$\begin{aligned}Z(\mathbf{d}) &= \sum_{i\in P} r_i d_i + \sum_{i\in N} r_i q_i + \mathbf{q}'(\mathbf{u} + \mathbf{v} - \mathbf{f} - \mathbf{g} - \mathbf{c}) + \mathbf{d}'(\mathbf{f} + \mathbf{g} - \mathbf{u} - \mathbf{v})\\ &= \mathbf{a}'\mathbf{d} + \mathbf{b}'\mathbf{q}. \quad \square\end{aligned}$$

We next give a necessary and sufficient condition for the existence of a finite performance ratio for an arbitrary online procurement strategy, and we provide an upper bound on the ratio if it exists. The following lemma is presented using the notation of Lemma 4.2.

Lemma 4.3. *For an arbitrary online strategy* $\mathbf{q} \geq \mathbf{0}$, *a finite performance ratio exists if and only if* $\mathbf{b}'\mathbf{q} \geq 0$. *Furthermore, if a finite ratio* ρ *exists, it satisfies*

$$\max_{1\leq i\leq n} \frac{-a_i}{r_i - c_i + \omega} \leq \rho \leq \max_{1\leq i\leq n} \frac{-a_i}{r_i - c_i},$$

where

$$\omega = \sum_{i=1}^{n} \max\{0, -(r_i - r_{i+1}) + (c_i - c_{i+1}) - s_i, r_i - (c_i - c_{i+1}) - h_i\}.$$

Proof of Lemma 4.3. Utilizing the lower bound in Lemmas 4.1 and 4.2, an upper bound for the performance ratio is

$$\rho \leq 1 + \sup_{\substack{d\geq 0\\ d\neq 0}} \frac{-\mathbf{a}'\mathbf{d} - \mathbf{b}'\mathbf{q}}{(\mathbf{r} - \mathbf{c})'\mathbf{d}}.$$

As the supremum in the upper bound is a linear-fractional program, it is equivalent to the following linear program:[1]

$$\left\{\begin{array}{ll}\max\limits_{y,z} & -\mathbf{a}'\mathbf{y}-\mathbf{b}'\mathbf{q}z\\ \text{s.t.} & (\mathbf{r}-\mathbf{c})'\mathbf{y}=1\\ & \mathbf{y}\geq\mathbf{0}\quad z\geq 0\end{array}\right\}\quad\text{whose dual is}\quad\left\{\begin{array}{ll}\min\limits_{\alpha} & \alpha\\ \text{s.t.} & \alpha(\mathbf{r}-\mathbf{c})\geq-\mathbf{a}\\ & 0\geq-\mathbf{b}'\mathbf{q}\end{array}\right\}.$$

Therefore, if the dual is feasible, a finite upper bound exists, which implies that a finite performance ratio exists. The dual is feasible when $\mathbf{b}'\mathbf{q}\geq 0$, which implies that the performance ratio is at most $1+\max_{1\leq i\leq n}\{-a_i/(r_i-c_i)\}$ (i.e., one plus the optimal dual value). This proves the sufficiency of $\mathbf{b}'\mathbf{q}\geq 0$ for a finite performance ratio. Repeating the above analysis with the upper bound from Lemma 4.1 provides the lower bound on the performance ratio and proves that $\mathbf{b}'\mathbf{q}\geq 0$ is also a necessary condition. □

4.2. Impossibility of a finite performance ratio for positive fixed ordering costs. We now consider the case where fixed ordering costs are positive, and we show the existence of a 2-period problem that does not admit a finite performance ratio.

LEMMA 4.4. *There exists a 2-period problem with $K_1, K_2>0$ that does not admit a finite performance ratio.*

PROOF OF LEMMA 4.4. We begin by modifying the proof of Lemma 4.3 to incorporate the fixed ordering costs K_i, $\forall\, i$. Indeed, we let $\mathscr{K}=\sum_{i=1}^n K_i\delta(q_i)$ and note that Lemma 4.2 is trivially modified to give $Z(\mathbf{d})=\mathbf{a}'\mathbf{d}+\mathbf{b}\mathbf{q}-\mathscr{K}$. Lemma 4.1 states that $Z^*(\mathbf{d})\leq(\mathbf{r}-\mathbf{c}+\omega\mathbf{e})'\mathbf{d}$, and we obtain the following lower bound for the performance ratio

$$\rho\geq 1+\sup_{\substack{d\geq 0\\ d\neq 0}}\frac{-\mathbf{a}'\mathbf{d}-\mathbf{b}'\mathbf{q}+\mathscr{K}}{(\mathbf{r}-\mathbf{c}+\omega\mathbf{e})'\mathbf{d}}.$$

As the supremum in the lower bound is a linear-fractional program, it is equivalent to the following linear program[2]

$$\left\{\begin{array}{ll}\max\limits_{y,z} & -\mathbf{a}'\mathbf{y}+(\mathscr{K}-\mathbf{b}'\mathbf{q})z\\ \text{s.t.} & (\mathbf{r}-\mathbf{c}+\omega\mathbf{e})'\mathbf{y}=1\\ & \mathbf{y}\geq\mathbf{0},\quad z\geq 0\end{array}\right\}\quad\text{whose dual is}\quad\left\{\begin{array}{ll}\min\limits_{\alpha} & \alpha\\ \text{s.t.} & \alpha(\mathbf{r}-\mathbf{c}+\omega\mathbf{e})\geq-\mathbf{a}\\ & 0\geq\mathscr{K}-\mathbf{b}'\mathbf{q}\end{array}\right\}.$$

Note that the dual is feasible when $\mathbf{b}'\mathbf{q}-\mathscr{K}\geq 0$. We next show that there exist cost and demand parameters that lead to an infeasible dual, which implies that the performance ratio is unbounded. We consider a two-period problem and analyze the periods sequentially.

For $n=1$, the online player does not know if a subsequent period exists, so he/she must assume the current period is the only period in deriving an ordering quantity that admits a finite performance ratio. We first consider a positive quantity $q_1>0$ which, for a finite performance ratio, requires $b_1q_1-K_1\geq 0$. If $q_1>d_1$, we have that $1\in P$, $b_1=-c_1-h_1$ and we require $(c_1+h_1)q_1+K_1\leq 0$, an impossibility. Because we can not eliminate the possibility that $q_1>d_1$, we conclude that $q_1=0$ and $1\in N$, which satisfies $b_1(0)-K_1\delta(0)=0$.

For $n=2$, $2\in P$ if $q_2>d_1+d_2$, which implies $b_2=-c_2-h_2-r_2$ and, for dual feasibility, we require $(c_2+h_2+r_2)q_2+K_2\leq 0$, again an impossibility. Therefore, we require $2\in N$; since we do not know d_2, we must account for the possibility that $d_2=0$, which implies $q_2\leq d_1$ to guarantee $2\in N$. Assuming this is the case, $b_2=r_2-c_2+s_2>0$ and, for dual feasibility, $q_2\geq K_2/(r_2-c_2+s_2)$. If $K_2/(r_2-c_2+s_2)>d_1$, then there is no possible value for q_2 to maintain a finite performance ratio. □

REMARK 4.1. Note that, for a single period, the first part of the proof of Lemma 4.4 can be modified slightly to obtain the proof of Theorem 3.1. Indeed, instead of using an upper bound on $Z^*(d)$, using the lower bound $Z^*(d)\geq(r-c)d-K$, which is tight because of Assumption 2.4, results in the original proof of Theorem 3.1.

[1] Technically, the vectors $\mathbf{a}$ and $\mathbf{b}$ are functions of the data $\mathbf{q}$ and variables $\mathbf{d}$ via the induced partition (P, N), so the optimization problem is actually *not* a linear program. However, a formal Lagrangian (weak) duality analysis arrives at the same conclusion and, for simplicity, we abuse notation and present the analysis as if $\mathbf{a}$ and $\mathbf{b}$ were constant vectors.

[2] See Footnote 1.

4.3. Avoiding positive inventory: A best-possible online strategy. We next show that, even if fixed ordering costs are zero in all periods, maintaining positive inventory precludes the existence of a finite performance ratio. In particular, we give a two-period example that illustrates this property.

LEMMA 4.5. *There exists a two-period problem with $K_1 = K_2 = 0$, where maintaining positive inventory implies that no finite performance ratio exists.*

PROOF OF LEMMA 4.5. We present a two-period ($n = 2$) example that does not allow a finite performance ratio. Suppose that inventory is held at the end of period 1; i.e., $I_1 = q_1 - d_1 > 0$ and $1 \in P$. If $2 \in P$, $b_1 = -c_1 - (h_1 + h_2)$, and $b_2 = -c_2 - h_2$. Applying Lemma 4.3, we see that a finite performance ratio exists if and only if $(c_1 + h_1 + h_2)q_1 + (c_2 + h_2)q_2 \le 0$, an impossibility. Finally, if $d_2 = 0$, then $2 \in P$ (i.e., the online player can not avoid seeing positive inventory at the end of period 2), and no finite performance ratio exists. □

Finally, we show that the simple and intuitive strategy of ordering the previous period's demand (i.e., $q_1 = 0$ and $q_i = d_{i-1}, i > 1$) is a best-possible online strategy. Essentially, this strategy only satisfies the backlogged demand from the previous period and does not attempt to guess future demands (or assumes they are zero). Given the confines of the complete lack of knowledge of future demand, this strategy is the best possible. However, the cost structure does provide us a performance guarantee for this strategy.

THEOREM 4.1. *If $K_i = 0$ for all i, the online procurement strategy $\mathbf{q} = \mathbf{d}^{-1}$ is a best-possible online strategy, with performance ratio*

$$1 + \max_{1\le i\le n} \frac{\tilde{s}_i}{r_i - c_i + \omega} \le \rho \le 1 + \max_{1\le i\le n} \frac{\tilde{s}_i}{r_i - c_i},$$

where

$$\omega = \sum_{i=1}^{n} \max\{0, -(r_i - r_{i+1}) + (c_i - c_{i+1}) - s_i, r_i - (c_i - c_{i+1}) - h_i\}.$$

PROOF OF THEOREM 4.1. The strategy $q_1 = 0$ and $q_i = d_{i-1}$, $i > 1$ has $i \in N$ for all i, which implies $a_i = -\tilde{s}_i$ and $b_i = r_i - c_i + \tilde{s}_i > 0$ for all i. Clearly, $\mathbf{b}'\mathbf{q} \ge 0$ and Lemma 4.3 states that a performance ratio exists, and it is at least

$$\max_{1\le i\le n} \frac{-a_i}{r_i - c_i + \omega} = \max_{1\le i\le n} \frac{\tilde{s}_i}{r_i - c_i + \omega},$$

and at most

$$\max_{1\le i\le n} \frac{-a_i}{r_i - c_i} = \max_{1\le i\le n} \frac{\tilde{s}_i}{r_i - c_i}.$$

Note that we do not know the exact performance ratio ρ; however, we can show it is a best-possible online strategy. To see this, we assume that there exists another strategy with a strictly smaller performance ratio. Lemmas 4.2 and 4.3 then imply that there must exist some period $i \in P$. However, the argument in the proof of Lemma 4.5 can be easily modified to show that no finite performance ratio exists for this conjectured strategy because $P \ne \varnothing$, and we have a contradiction. □

4.4. The asymptotic case. In this section, we consider a special cost structure that introduces discounting and allows us to make statements about the performance ratio of the online strategy $\mathbf{q} = \mathbf{d}^{-1}$ for an infinite number of periods. In particular, we assume that there exist $r, c, s, h > 0$ and $\delta \in (0, 1)$ such that

$$r_i = r\delta^i, \qquad c_i = c\delta^i, \qquad s_i = s\delta^i, \qquad h_i = h\delta^i, \quad \text{for } i = 1, \ldots, n.$$

THEOREM 4.2. *If $K_i = 0$ for all i, the online procurement strategy $\mathbf{q} = \mathbf{d}^{-1}$ is a best-possible online strategy, with a performance ratio*

$$1 + \frac{s}{r - c + \bar{\omega}} \frac{1}{1-\delta} \le \rho \le 1 + \frac{s}{r - c} \frac{1}{1-\delta},$$

where

$$\bar{\omega} = \frac{\max\{0, r - c(1-\delta) - h\}}{1-\delta}.$$

PROOF OF THEOREM 4.2. We augment the proof of Theorem 4.1 to prove this result. Indeed, for the upper bound on the performance ratio, we have that

$$\max_{1\le i\le n} \frac{\tilde{s}_i}{r_i - c_i} = \max_{1\le i\le n} \frac{s\sum_{j=0}^{n-i}\delta^j}{r - c} = \max_{1\le i\le n} \frac{s((1-\delta^{n-i+1})/(1-\delta))}{r - c} = \frac{s}{r - c}\frac{1-\delta^n}{1-\delta}.$$

Taking the limit $n \to \infty$ proves the upper bound. Next, we consider the lower bound; we have that

$$\begin{aligned}
\omega &= \sum_{i=1}^{n} \max\{0, -(r_i - r_{i+1}) + (c_i - c_{i+1}) - s_i, r_i - (c_i - c_{i+1}) - h_i\} \\
&= \sum_{i=1}^{n} \max\{0, -(r\delta^i - r\delta^{i+1}) + (c\delta^i - c\delta^{i+1}) - s\delta^i, r\delta^i - (c\delta^i - c\delta^{i+1}) - h\delta^i\} \\
&= \sum_{i=1}^{n} \max\{0, -(r-c)\delta^i(1-\delta) - s\delta^i, r\delta^i - c\delta^i(1-\delta) - h\delta^i\} \\
&= \max\{0, r - c(1-\delta) - h\} \sum_{i=1}^{n} \delta^i \\
&= \max\{0, r - c(1-\delta) - h\} \frac{\delta - \delta^{n+1}}{1-\delta}.
\end{aligned}$$

Consequently, for the lower bound on the performance ratio, we have that

$$\max_{1 \le i \le n} \frac{\tilde{s}_i}{r_i - c_i + \omega} = \max_{1 \le i \le n} \frac{s\sum_{j=0}^{n-i} \delta^j}{r - c + \omega\delta^{-i}} = \max_{1 \le i \le n} \frac{s((1-\delta^{n-i+1})/(1-\delta))}{r - c + \omega\delta^{-i}} = \frac{s}{r - c + \omega\delta^{-1}} \frac{1-\delta^n}{1-\delta}.$$

Letting $\bar{\omega} = \omega/\delta$ and taking the limit $n \to \infty$ proves the lower bound. □

5. Future research. In this section we briefly mention potential directions for continuing the research in this paper. The first direction considers hybrids of the two problems considered herein. The second concerns randomized online algorithms.

This paper considers two main problems: (1) perishable products with lost sales and (2) durable products with backlogged demand. An interesting direction of research would be to consider mixtures of these two problems, such as (A) durable products with lost sales and (B) perishable products with backlogged demand. Practically speaking, Problem (A) is likely more relevant.

Another direction that can be studied is *randomized* online algorithms. The competitive ratio might be relevant if randomized algorithms are considered, as trivialities can be averaged out. Also, regarding the durable products with backlogged demand case, randomized online algorithms might be able to overcome some of the limitations proved in §4 for deterministic online algorithms. Finally, randomization can reduce the conservatism of the online algorithms and increase their practicality.

Appendix A. Proof of Lemma 4.1

Proof. Because ordering d_i in period i is a feasible offline solution, we have the lower bound $Z^*(\mathbf{d}) \ge (\mathbf{r} - \mathbf{c})'\mathbf{d} > 0$ for $K_i = 0$ for all i; the lower bound is positive because $\mathbf{r} - \mathbf{c} > \mathbf{0}$ and $\mathbf{d} \ne \mathbf{0}$. We next derive the upper bound, for arbitrary fixed ordering costs K_i, using linear programming duality theory.

The offline problem (3) is not a convex optimization problem, so we therefore study the formulation using binary integer programming, which we then relax to obtain a linear program, whose dual provides a valuable upper bound on $Z^*(\mathbf{d})$.

We define variables $p_i \ge 0$ and $n_i \ge 0$ as the positive and negative parts of I_i, for $i = 1, \ldots, n$. In other words, $I_i = p_i - n_i$ and $|I_i| = p_i + n_i$; additionally, for convenience, we define $p_0 = n_0 = 0$. We also let w_i, $i = 1, \ldots, n$ denote binary variables that indicate whether I_i is nonnegative or nonpositive. Letting $D = \sum_{i=1}^{n} d_i$ and observing that $|I_i| \le D$, the relevant binary constraints are $p_i \le w_i D$ and $n_i \le (1 - w_i)D$ for $i = 1, \ldots, n$. We ignore the fixed ordering costs K_i to find an integer programming upper bound on $Z^*(\mathbf{d})$:

$$\begin{aligned}
\max_{\substack{x_i, n_i, p_i, q_i, w_i \\ i=1,\ldots,n}} \quad & \sum_{i=1}^{n} r_i x_i - s_i n_i - h_i p_i - c_i q_i, \\
\text{s.t.} \quad & x_i \le q_i + p_{i-1}, \quad i = 1, \ldots, n, \\
& x_i \le d_i + n_{i-1}, \quad i = 1, \ldots, n, \\
& p_i \le w_i D, \quad i = 1, \ldots, n, \\
& n_i \le (1 - w_i)D, \quad i = 1, \ldots, n,
\end{aligned}$$

$$\sum_{j=1}^{i}(q_j - d_j) = p_i - n_i, \quad i = 1, \ldots, n,$$

$$n_i, p_i, q_i \geq 0, \;\; x_i \text{ free}, \;\; w_i \in \{0, 1\}, \quad i = 1, \ldots, n.$$

We next change variables by using $\hat{q}_i = \sum_{j=1}^{i} q_j$, $\hat{q}_0 = 0$, $\hat{d}_i = \sum_{j=1}^{i} d_j$, and $\hat{d}_0 = 0$; this change of variable requires that we add the constraints $\hat{q}_i - \hat{q}_{i-1} \geq 0$ for all i to replace the nonnegativity constraint on the q_i. Additionally, note that

$$\mathbf{c}'\mathbf{q} = \sum_{i=1}^{n} c_i q_i = \sum_{i=1}^{n} c_i(\hat{q}_i - \hat{q}_{i-1}) = \hat{\mathbf{q}}'\mathbf{c} - \hat{\mathbf{q}}'\mathbf{c}^{+1}.$$

Therefore, by relaxing the integer constraints, we obtain the following linear programming upper bound in matrix form, with associated dual variables:

$$\begin{aligned}
\max_{x, n, p, \hat{q}, w} \quad & \mathbf{r}'\mathbf{x} - \mathbf{s}'\mathbf{n} - \mathbf{h}'\mathbf{p} - (\mathbf{c} - \mathbf{c}^{+1})'\hat{\mathbf{q}}, & \\
\text{s.t.} \quad & \mathbf{x} - \hat{\mathbf{q}} + \hat{\mathbf{q}}^{-1} - \mathbf{p}^{-1} \leq \mathbf{0} & : \boldsymbol{\alpha}, \\
& \mathbf{x} - \mathbf{n}^{-1} \leq \mathbf{d} & : \boldsymbol{\beta}, \\
& \mathbf{p} - \mathbf{w}D \leq \mathbf{0} & : \boldsymbol{\gamma}, \\
& \mathbf{n} + \mathbf{w}D \leq D\mathbf{e} & : \boldsymbol{\delta}, \\
& \hat{q} - \mathbf{p} + \mathbf{n} = \hat{\mathbf{d}} & : \boldsymbol{\theta}, \\
& \mathbf{w} \leq \mathbf{e} & : \boldsymbol{\phi}, \\
& \hat{\mathbf{q}}^{-1} - \hat{\mathbf{q}} \leq \mathbf{0} & : \boldsymbol{\sigma}, \\
& \mathbf{n}, \mathbf{p}, \mathbf{w} \geq \mathbf{0}, \quad \mathbf{x}, \hat{\mathbf{q}} \text{ free.} &
\end{aligned}$$

The dual of this linear program is

$$\begin{aligned}
\min_{\alpha, \beta, \gamma, \delta, \theta, \phi, \sigma} \quad & \boldsymbol{\beta}'\mathbf{d} + D\boldsymbol{\delta}'\mathbf{e} + \boldsymbol{\theta}'\hat{\mathbf{d}} + \boldsymbol{\phi}'\mathbf{e}, \\
\text{s.t.} \quad & \boldsymbol{\alpha} + \boldsymbol{\beta} = \mathbf{r}, \\
& \boldsymbol{\beta}^{+1} - \boldsymbol{\delta} - \boldsymbol{\theta} \leq \mathbf{s}, \\
& \boldsymbol{\alpha}^{+1} - \boldsymbol{\gamma} + \boldsymbol{\theta} \leq \mathbf{h}, \\
& \boldsymbol{\alpha} - \boldsymbol{\alpha}^{+1} - \boldsymbol{\theta} - \boldsymbol{\sigma}^{+1} + \boldsymbol{\sigma} = (\mathbf{c} - \mathbf{c}^{+1}), \\
& D(\boldsymbol{\delta} - \boldsymbol{\gamma}) + \boldsymbol{\phi} \geq \mathbf{0}, \\
& \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\phi}, \boldsymbol{\sigma} \geq \mathbf{0}, \quad \boldsymbol{\theta} \text{ free.}
\end{aligned}$$

Now, consider the following dual feasible solution:

$$\begin{gathered}
\boldsymbol{\alpha} = \mathbf{r}, \\
\boldsymbol{\beta} = \mathbf{0}, \\
\boldsymbol{\gamma} = \max\{\boldsymbol{\delta}, \mathbf{r} - (\mathbf{c} - \mathbf{c}^{+1}) - \mathbf{h}\}, \\
\boldsymbol{\delta} = \max\{\mathbf{0}, -\boldsymbol{\theta} - \mathbf{s}\}, \\
\boldsymbol{\theta} = (\mathbf{r} - \mathbf{r}^{+1}) - (\mathbf{c} - \mathbf{c}^{+1}), \\
\boldsymbol{\phi} = D(\boldsymbol{\gamma} - \boldsymbol{\delta}), \\
\boldsymbol{\sigma} = \mathbf{0},
\end{gathered}$$

whose cost,

$$\begin{aligned}
D\boldsymbol{\delta}'\mathbf{e} + \boldsymbol{\theta}'\hat{\mathbf{d}} + \boldsymbol{\phi}'\mathbf{e} &= D\boldsymbol{\gamma}'\mathbf{e} + \theta'\hat{\mathbf{d}}, \\
&= (\mathbf{d}'\mathbf{e})(\boldsymbol{\gamma}'\mathbf{e}) + \tilde{\boldsymbol{\theta}}'\mathbf{d}
\end{aligned}$$

$$= (\mathbf{d}'\mathbf{e})(\boldsymbol{\gamma}'\mathbf{e}) + \sum_{i=1}^{n} d_i \sum_{j=i}^{n} \theta_j$$

$$= (\mathbf{d}'\mathbf{e})(\boldsymbol{\gamma}'\mathbf{e}) + \sum_{i=1}^{n} d_i \sum_{j=i}^{n} (r_i - c_i - (r_{i+1} - c_{i+1}))$$

$$= (\mathbf{d}'\mathbf{e})(\boldsymbol{\gamma}'\mathbf{e}) + \sum_{i=1}^{n} d_i (r_i - c_i)$$

$$= ((\boldsymbol{\gamma}'\mathbf{e})\mathbf{e} + \mathbf{r} - \mathbf{c})'\mathbf{d}$$

is an upper bound on $Z^*(\mathbf{d})$. Defining $\omega = \boldsymbol{\gamma}'\mathbf{e}$ completes the proof. □

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