Crowdfunding via Revenue-Sharing Contracts

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Abstract. *Problem definition*: We analyze a new model of crowdfunding recently introduced by Bolstr, Localstake, and Startwise. A platform acts as a matchmaker between a firm needing funds and a crowd of investors willing to provide capital. After the firm is funded, it pays back the investors using revenue-sharing contracts, with a prespecified investment multiple (investors will receive $M \geq 1$ dollars for every dollar invested) and a revenue-sharing proportion, over an investment horizon of uncertain duration. *Academic/practical relevance*: We analyze the revenue-sharing contract approach to crowdfunding, and we assist the firm to determine its optimal contract parameters to maximize its expected net present value (NPV) subject to investor participation constraints and platform fees. *Methodology*: A natural multiperiod formulation for the firm's problem results in an intractable stochastic optimization model, which we approximate using a deterministic model. In the approximation model, we use a cash buffer for dealing with cash flow uncertainties; we are able to solve the approximation model analytically. *Results*: Parametrized on real data from Bolstr campaigns, our approximation solutions give an NPV in the stochastic problem that is within 0.2% of the simulation-based optimal NPV for all levels of cash flow uncertainty. We compare revenue-sharing contracts with equity crowdfunding and observe that the former result in higher NPVs and comparable bankruptcy probabilities. We also compare revenue-sharing contracts with fixed rate loans and find that, for most cases considered, revenue-sharing contracts provide a higher NPV and a lower probability of bankruptcy than a fixed rate loan. We also show that these benefits are more significant for firms with higher levels of cash flow uncertainty. *Managerial implications*: Revenue-sharing contracts are a novel approach to crowdfunding, and we show that are superior to other financing models.

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1. Introduction

Crowdfunding is a relatively new approach for firms to raise capital from a crowd of individuals rather than traditional sources of capital (e.g., banks, venture capitalists, etc.). The crowdfunding industry is already large and growing fast. In 2013, the industry was estimated to have raised over $5.1 billion worldwide (Noyes 2014). Looking to the future, PricewaterhouseCoopers (2015) has approximated that, by 2025, crowdfunding will be a $150 billion industry. A recent *Wall Street Journal* article (Karabell 2015) provides some insight for this growth: (1) an individual crowdfunding investment can be risky, offering high returns; (2) lenders can diversify their risk by spreading their investments over different crowdfunding campaigns; and (3) because of recent financial crises, investors have decreased trust in traditional financial investments (e.g., banks, stocks, etc.), and therefore, nontraditional lending markets have increased appeal. A recent regulatory change has also reduced the barriers to entry: starting from May 2016, ordinary people are permitted to invest in small firms (U.S. Securities and Exchange Commission 2016); before May 2016, only accredited investors (i.e., those with an annual income of at least $200,000 or a net worth of at least $1 million) could invest (Cowley 2016). This new crowdfunding regulation, which allows firms to raise up to $1 million over a 12-month period, provides a great opportunity for firms to raise their investment targets more easily.

A number of different firms have become rather well known: Kickstarter, Indiegogo, GoFundMe, Kiva, Prosper, and Lending Club. These firms have vastly different crowdfunding models driving their businesses. For example, Kickstarter and Indiegogo, perhaps the most well-known crowdfunding firms, essentially solicit donations for individuals and firms needing capital. Kiva deals with microloans, targeting low-income entrepreneurs worldwide, that must be repaid to the lender. Prosper and Lending Club operate in the peer-to-peer lending marketplace, where loans have fixed repayment terms. The peer-to-peer lending market was the most dominant alternative finance market in 2015 in the United States, with approximately $25.7 billion raised (Alois 2016).

Our paper is motivated by an emergent model of crowdfunding pioneered by Bolstr (www.bolstr.com), Localstake (www.localstake.com), and Startwise (www.startwise.com), which targets small- and medium-sized firms needing capital; however, this model is not intrinsically limited by firm size and could be applied to any sized firm. These platforms match a firm needing funding with investors from the crowd, and the firm pays back the investors via *revenue-sharing contracts*. This more flexible repayment agreement is linked with the financial performance of the firm, allowing variable payments and investment horizons and thus, reducing financial stress on the borrower. In contrast, if business goes well, the borrower is obligated to increase the payments, thus reducing the investment horizon, which results in a higher effective interest rate for the investors. Therefore, revenue-sharing contracts intuitively align a firm's and investors' incentives in a way not possible with traditional fixed rate loans. Notably, many of the loans administered by, for example, Bolstr are repaid well before their estimated investment horizon: according to the Bolstr website, the estimated investment horizons usually range from 2 to 5 years for loan sizes of $25,000–$500,000, but JP Morgan Chase Company (2015) reports on the example of a lobster roll restaurant in Chicago that paid back its loan of $70,000 in 7 months. Furthermore, in 2016, Bolstr announced that, "An investor who participated at the minimum investment level in every deal would have a portfolio tracking to a 19.18% net Internal Rate of Return."

The speed of funding in crowdfunding can be very fast with respect to traditional funding sources. Traditional loans, such as from banks and the U.S. Small Business Administration (SBA), have low approval rates and typically take 2–3 months to fund. However, revenue-sharing contracts can fund very fast in practice. The following quotes are real subject lines from marketing emails from Bolstr:

"Paddy Wagon Raised $20,000 in Less than 10 Minutes,"

"Dubina Brewing Co. Raised $30,000 in 20 minutes,"

"The Bacon Jams Raised $40,000 in Less than 30 Minutes," and

"Underground Butcher Raised $75,000 in Less Than 1 Hour."

Firms can alternatively raise investments from other online lenders, such as Lending Club, Prosper, and Kabbage, but according to the Bolstr website, loans from these lenders have annual percentage rates as high as 80%, whereas revenue-sharing loans have annual percentage rates of 8%–25%. Therefore, revenue-sharing contracts have advantages over traditional and alternative funding sources, such as having flexible payments, fast funding time, and lower equivalent interest rates. In our paper, we show that the firm's net present value (NPV) under a revenue-sharing contract is larger than that of equity crowdfunding and that the firm's probabilities of bankruptcy are comparable. Additionally, in most cases considered, a firm's NPV under a revenue-sharing contract is larger than the NPV of a fixed rate loan, even when the latter has low annual interest rates. Similarly, a firm's probability of bankruptcy is lower under a revenue-sharing contract than under a fixed rate loan. These benefits stem from the flexible nature of the revenue-sharing contract.

The proposed revenue-sharing contract bears some superficial similarity to performance-sensitive debt (e.g., step-up bonds and performance-pricing loans) studied by Manso et al. (2010), where the debt payments depend on the borrower's performance: the borrower pays higher interest rates during low performance and lower interest rates during high performance. However, this approach has the opposite behavior of our proposed revenue-sharing contract, because under the latter, a high-performing firm will have high debt payments and will pay off the fixed loan amount early, resulting in a higher effective interest rate. Performance-sensitive debt has been shown to harm both the borrower and the investors via earlier borrower default (Manso et al. 2010), whereas our new model can result in positive outcomes for all parties.

In this paper, we provide an analysis of the firm's multiperiod problem of maximizing its expected NPV, subject to platform fees and investor participation constraints, for stochastic revenues and costs. The firm decides how much investment $Y \geq 0$ is needed, an investment multiple $M \geq 1$ (where investors are guaranteed to be paid back M times their initial investment), and a revenue-sharing proportion $\gamma \geq 0$ (in each period, the firm pays all investors a proportion γ of its revenues). These variables induce a stochastic investment horizon T and a stochastic bankruptcy time B (possibly infinite). The platform fees consist of an origination percentage $\alpha \in [0, 1]$ (where the firm pays a fee αY to the platform at time 0) and a servicing percentage $\beta \in [0, 1]$ (where in each period, the firm pays a β percent of all investor revenue payments to the platform). We design a stochastic programming formulation of the firm's problem, which is a rather intractable model.

Because the stochastic model is difficult to analyze, we derive a deterministic approximation model for it, in which we use a cash buffer to cope with uncertainties. We then solve the approximation problem analytically, which provides generalizable insights. We also solve our stochastic model numerically using Monte Carlo simulation and a grid-based optimization framework for serially correlated random cash flows that are parameterized using real data from Bolstr. We then compare the NPVs of the approximate and optimal solutions in the true stochastic model. We conclude that our approximation provides high-quality solutions: the worst

case average error of the approximation solution's NPV over all feasible Bolstr campaigns for all levels of cash flow uncertainty is approximately 0.2%.

Finally, we compare the performance of the proposed revenue-sharing contract with equity crowdfunding and fixed rate loans, and we identify which type of financing results in a higher NPV or a lower chance of bankruptcy for a firm; we find that the revenue-sharing contract is superior in most cases. We next provide a literature review to appropriately position our paper's contributions.

1.1. Literature Review

There are many papers that study other crowdfunding models and papers that study various aspects of the interface of operations and finance as well as a vast literature on revenue-sharing contracts, primarily in the areas of supply chain management and educational financing. In this section, we survey the most relevant results in these three streams. However, to the best of our knowledge, there are no studies investigating crowdfunding via revenue-sharing contracts, except Fatehi and Wagner (2017), which is a book chapter summary of a preliminary version of this research. Thus, our research is uniquely at the intersection of these two literature streams.

Babich et al. (2017) provided a detailed study on how crowdfunding impacts the financing decisions of entrepreneurs, banks, and venture capital investors. Chen et al. (2017) considered fundraising success in crowdfunding and studied the optimal referring policies by entrepreneurs empirically and analytically. Chakraborty and Swinney (2016) studied how entrepreneurs can signal their product's quality to contributors via campaign design in reward-based crowdfunding. Zhang and Liu (2012) studied a data set from Prosper, focusing on a study of herding behavior in microloan markets. Lin and Viswanathan (2015) also analyzed data from Prosper and studied the "home bias" in crowdfunding markets. Iyer et al. (2015) studied a data set from Prosper and showed that lenders use standard information along with nonstandard, or soft, information to evaluate borrower creditworthiness. Belleflamme et al. (2014) compared two forms of crowdfunding, profit sharing and preordering of products, and showed that, if the firm's investment goal is relatively high with respect to the market size, the firm prefers profit-sharing crowdfunding. Finally, Wei and Lin (2016) study, both theoretically and empirically, the difference between auctions and posted prices on Prosper.com.

Our paper is also relevant to the literature on the interface of operations and finance. In particular, our research is relevant to operational financing. There can be many sources of financing, including traditional bank financing, as well as more novel arrangements that have been attracting attention in the operations management literature, such as buyer financing (Deng et al. 2018),

supplier trade credits (Lee et al. 2017), or both (Kouvelis and Zhao 2017). We identify crowdfunding as an additional source of financing, and although we focus on revenue-sharing crowdfunding, we also consider equity crowdfunding.

The structure of our crowdfunding contracts, revenue sharing, has also been extensively studied in the supply chain management literature. However, the combination with financial aspects is limited. Kouvelis and Zhao (2015) studied contract design for a supply chain with one supplier and one retailer in the presence of financial constraints and bankruptcy costs, and they show that a revenue-sharing contract can still coordinate the supply chain. Kouvelis et al. (2017) showed that revenue-sharing contracts can achieve high efficiency in the presence of cost uncertainty and working capital constraints. Similarly, we show that a firm can benefit significantly from revenue-sharing contracts under stochastic cash flows.

Revenue-sharing contracts have also appeared in the education field as a novel approach to funding students. Nerlove (1975) studied an income-contingent loan program for the financing of education, which was originally proposed by Friedman and Kuznets (1945) for professional education and by Friedman (1955) for vocational education. One example of such a loan program is the Yale Tuition Postponement Option, which started in 1971 but was discontinued in 1978 (Ladine 2001). Another example is the Pay-It-Forward plan in Oregon: in 2013, state legislators proposed a program in which students could attend public colleges without paying tuition and in return, would pay 3% of their future income to the state for several years after graduation (Palacios and Kelly 2014). Similarly, the Back-a-Boiler program at Purdue University, which started in the fall of 2016 and has already raised $2.2 million, provides funds to undergraduate students to finance their education through an Income Share Agreement, in which students agree to pay a percentage of their future income over a standard payment term (Purdue 2016). The results and insights of our paper can also be applied to these student loan agreements.

1.2. Contributions

The contributions of our paper are as follows.

1. We are the first, to our knowledge, to study a new emergent model of crowdfunding pioneered by Bolstr, Localstake, and Startwise. Furthermore, our models are parameterized using real data from 56 Bolstr campaigns.

2. We study a firm's expected NPV maximization problem under a revenue-sharing contract, which is an intractable stochastic model. To overcome the technical difficulties, we design a tractable deterministic approximation model, in which we use a cash buffer to cope with cash flow uncertainties. We solve the approximation model analytically, which provides qualitative insights. Numerical experiments, calibrated on

real Bolstr data, indicate that the approximation solutions, when inserted into the true stochastic model, result in an expected NPV that is within 0.2% of the true optimal NPV on average. Furthermore, the approximation solutions result in almost the same bankruptcy probabilities as the optimal stochastic solutions.

3. Our results provide managerial guidelines for the firm.

(a) A firm can attain a higher NPV and a comparable probability of bankruptcy under a revenue-sharing contract than under equity crowdfunding. We show that the NPV benefit of revenue sharing increases as the firm's cash flow volatility increases.

(b) A firm can attain a higher NPV and a lower probability of bankruptcy under a revenue-sharing contract than under a more traditional fixed rate loan. We also show that these benefits are more significant for firms with higher levels of cash flow uncertainty. Intuitively, these benefits are because of the more flexible nature of the revenue-sharing contract.

(c) As cash flow uncertainty increases, the optimal investment amount increases, and the revenue-sharing percentage decreases, resulting in a stochastically larger investment horizon (e.g., larger mean and variance). However, the firm's maximized NPV is rather insensitive to cash flow uncertainty. Thus, although the details of the optimal revenue-sharing contract can change considerably with cash flow uncertainty, the bottom line NPV is rather robust to this uncertainty.

2. Stochastic Model of Firm

In this section, we detail our basic stochastic model of a firm with revenue and cost in time period t, $R_t \geq 0$ and $C_t \geq 0$, that are random variables; we model these cash flows using random walks with drift, which are explained in detail in Section 2.2. Many firms on Bolstr, Localstake, and Starwise have raised money to upgrade their existing stores, build a store in a new location, buy new equipment, and create/upgrade their websites. Because of this expansion, the firm has cash flow shortages for a limited time and therefore, needs to raise capital in the amount of $Y \geq 0$ (e.g., dollars) via an intermediary platform that pairs interested individual investors that are willing to invest. Therefore, the investment Y is a buffer to avoid a negative cash flow.

The investors do not receive equity in the firm but are paid back, with interest, via a revenue-sharing contract: at the end of each time period t, the firm is contractually obligated to pay out to all investors a proportion $\gamma \geq 0$ of its revenues for that time period. These payments continue until each investor receives a multiple $M \geq 1$ of his/her initial investment, which occurs at time $t = T$ (for all investors); this definition of M is motivated by practical implementations (e.g., Bolstr, Localstake, and Startwise). Note that the investment payments are not fixed, because they depend on firm revenues, which can

vary. The contract's duration is, therefore, the stochastic stopping time

$$T = \min\left\{\hat{T} \geq 1 : \sum_{t=1}^{\hat{T}} \gamma R_t \geq MY\right\}, \tag{1}$$

which captures the firm's contractual obligation to pay γ percent of its revenue to investors until a total nominal amount of MY has been paid. Time period $t = 0$ is the initialization of the revenue-sharing contract when the firm receives total investment Y. The firm must also pay the platform (1) an origination fee of $\alpha \in [0, 1]$ percent of the total amount raised Y at time $t = 0$ and (2) a servicing fee of $\beta \in [0, 1]$ percent of all revenue payments made to investors at times $t > 0$.

We next discuss cash flows, and for simplicity, we assume that the risk-free interest rate is zero (i.e., cash does not earn interest). If, in period t, $R_t - C_t < 0$, there is a cash shortfall; however, this can potentially be addressed using an excess of cash from previous periods. Thus, we focus on cumulative cash flows. If there exists $\tau \geq 1$, such that $\sum_{t=1}^{\tau}(R_t - C_t) < 0$, then the firm needs cash in month τ. The firm's cumulative cash flow at the end of month $\tau = 1, \ldots, T$, during the revenue-sharing contract, is $\sum_{t=1}^{\tau}(R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma\sum_{t=1}^{\tau} R_t$; if $\tau > T$, after the completion of the contract, the cumulative cash flow is $\sum_{t=1}^{\tau}(R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma\sum_{t=1}^{T} R_t$. These two scenarios can be combined into one expression for the cash flow in period $\tau \geq 1$: $\sum_{t=1}^{\tau}(R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma\sum_{t=1}^{\min\{\tau,T\}} R_t$.

Because of the stochasticity of the firm's cash flows, the firm can potentially go bankrupt for any combination of contractual parameters (Y, M, γ). Babich and Tang (2016) and Uhrig-Homburg (2005) model bankruptcy as occurring during the first period when the firm is cash flow negative; we adopt this approach. Letting B denote the time period when the firm goes bankrupt, we model B as a stochastic stopping time (that depends on the stochastic stopping time T):

$$B = \min\left\{\hat{B} \geq 1 : \sum_{t=1}^{\hat{B}}(R_t - C_t) + (1 - \alpha)Y \right.$$
$$\left. - (\beta + 1)\gamma \sum_{t=1}^{\min\{\hat{B},T\}} R_t < 0\right\}. \tag{2}$$

The platform and investors receive their payments at the end of each month if the borrower is not bankrupt; thus, the revenue-sharing contract is in effect for $\tau = 1, \ldots, \min\{B, T\}$. Because costs and revenues are stochastic, a risk-neutral firm wants to maximize the firm's expected NPV:

$$E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1 + r_t)^t}\right] - (\beta + 1)\gamma E\left[\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1 + r_t)^t}\right] + (1 - \alpha)Y, \tag{3}$$

where $\gamma E\left[\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t}\right]$ is the expected NPV of all payments made to investors, $\beta\gamma E\left[\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t}\right]$ is the expected NPV of all servicing fees paid to the platform, αY is the origination fee paid to the platform at time $t = 0$, and in period t, risk is quantified via imputed discount rates r_t (Arrow and Lind 1978). We assume that these discount rates are known. According to Pratt et al. (2014), small-sized firms can estimate their NPV by using the cost of capital as the discount rate. Alternatively, Wei and Lin (2016) assume that firms determine their discount rate according to the lowest interest rate offered to them from other financial institutions.

Finally, we consider the investors. We assume that there is a large pool of investors and that the target investment is raised by n investors, where investor i invests an amount $y_i \in (0, Y]$ and $\sum_{i=1}^n y_i = Y$. We model the investor participation constraints as

$$E\left[\sum_{t=1}^{\min\{B,T\}} \frac{y_i}{Y} \frac{\gamma R_t}{(1+\delta_t)^t} - y_i\right] \geq A_i, \ i = 1, \ldots, n, \quad (4)$$

where $\frac{y_i}{Y}$ is the fraction of the total payment γR_t that investor i receives in period t, δ_t is the discount rate of investors in period t, the left-hand side is the investor's expected NPV, and $A_i \geq 0$ is the target rate of return, in NPV terms, for investor i. In other words, A_i captures investor i's opportunity cost of alternative investments. We simplify the constraints in Expression (4) to the following single constraint:

$$E\left[\sum_{t=1}^{\min\{B,T\}} \frac{\gamma}{Y} \frac{R_t}{(1+\delta_t)^t}\right] \geq \max_{1\leq i\leq n}\left\{\frac{A_i}{y_i}\right\} + 1. \quad (5)$$

The firm is able to select the investment amount Y, the multiple M, and the revenue-sharing proportion γ to maximize its expected NPV. The above analysis results in the firm's problem:

$$z_F = \max_{Y,M,\gamma} E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t}\right] - (\beta+1)\gamma E\left[\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t}\right]$$
$$+ (1-\alpha)Y$$

subject to (s.t.) $T = \min\left\{\hat{T} \geq 1 : \sum_{t=1}^{\hat{T}} \gamma R_t \geq MY\right\}$
(contractual obligation)

$$B = \min\left\{\hat{B} \geq 1 : \sum_{t=1}^{\hat{B}} (R_t - C_t) + (1-\alpha)Y\right.$$
$$\left. - (\beta+1)\gamma \sum_{t=1}^{\min\{\hat{B},T\}} R_t < 0\right\}$$
(bankruptcy definition)

$$E\left[\sum_{t=1}^{\min\{B,T\}} \frac{\gamma}{Y} \frac{R_t}{(1+\delta_t)^t}\right] \geq \max_{1\leq i\leq n}\left\{\frac{A_i}{y_i}\right\} + 1$$

$Y, \gamma \geq 0, M \geq 1.$
(investor participation)

(6)

2.1. Model Parameterization Using Data from Bolstr.com

We have collected cost and revenue projections from 56 campaigns on the Bolstr platform and performed regression analyses on them. The R^2 values for the cost regressions ranged from 0.595 to 0.995, with a mean of 0.905 and a standard deviation of 0.079. The R^2 values for the revenue regressions ranged from 0.725 to 0.995, with a mean of 0.922 and a standard deviation of 0.065. Therefore, the Bolstr data suggest that linear models of cost and revenue projections, in expectation, are reasonable assumptions.

We denote a and b as the intercept and slope of a generic revenue regression line, respectively; similarly, we denote c and d as the intercept and slope of an arbitrary cost regression line, respectively. Letting $E[R_t]$ and $E[C_t]$ denote the expected revenue and cost in period t, respectively, we assign

$$E[R_t] = a + bt \quad \text{and} \quad E[C_t] = c + dt. \quad (7)$$

Many, but not all, of our results will utilize the linearity of cash flows.

2.2. Modeling Cash Flows

Motivated by the cash flow models in Dechow et al. (1998), we generate revenues $(R_1, R_2, \ldots)$ and costs $(C_1, C_2, \ldots)$ using random walk processes:

$$R_t = R_{t-1} + Z_t^r \quad \text{and} \quad C_t = C_{t-1} + Z_t^c, \quad t \geq 1, \quad (8)$$

where $R_0 = a$ and $C_0 = c$ are given in Equation (7). Z_t^r are independent normal random variables with common mean $\mu^r = b$, where b is given by Equation (7), and standard deviation $\sigma^r = \mu^r/k$, where k is a tunable parameter. Similarly, Z_t^c are independent normal random variables with common mean $\mu^c = d$, where d is given by Equation (7), and standard deviation $\sigma^c = \mu^c/k$. The means can be easily calculated: $E[R_t] = a + bt$ and $E[C_t] = c + dt$, in agreement with Equation (7). Furthermore, the random walk model exhibits serial correlation: it is straightforward to show that, for $s < t$, $cov(R_s, R_t) = s(\sigma^r)^2$ and $cov(C_s, C_t) = s(\sigma^c)^2$. Note that Brownian motion is a limit of a random walk process (Kac 1947). Therefore, our proposed random walk models for revenues and costs can be considered as approximate Brownian motion processes with drifts μ^r and μ^c and volatilities σ^r and σ^c, respectively (Ross 1996, Sigman 2006).

2.3. Analysis Roadmap

We found an analytical solution to Problem (6) to be intractable. In the next section, we derive an approximation for the stochastic problem. The approximation problem is a deterministic relaxation where $\sigma^r = \sigma^c = 0$, but we add a cash flow buffer to the bankruptcy definition to deal with cash flow uncertainties in the

stochastic model; in this case, we are able to derive analytical solutions that provide generalizable insights. These results are provided in Section 3. To evaluate the quality of our approximation, we also solve Problem (6) for real Bolstr data using Monte Carlo simulation to determine the stochastic stopping times (B, T) and expectations combined with numerical optimization on a fine grid of (Y, M, γ) space. We utilize random walks with drift for the cash flows, where $\sigma^r = \mu^r / k$ and $\sigma^c = \mu^c / k$ for various values of k; this analysis can be found in Section 4. Finally, in Sections 5 and 6, we show that the revenue-sharing contract compares favorably with equity crowdfunding and fixed rate loans, respectively.

3. Deterministic Approximation to Stochastic Model

In this section, we consider a deterministic approximation to Problem (6), where the cash flows R_t and C_t are known exactly. This simplification results in the conversion of the bankruptcy time B and investment duration T into parameters rather than random variables. We assume that, given that cash flows are known exactly, the firm desires to avoid bankruptcy. Thus, reversing the condition for bankruptcy in Equation (2), we introduce constraints that require the firm to be cash flow positive above $\theta \geq 0$ for all time periods $\tau \geq 1$, where θ is a cash buffer to account for cash flow uncertainties in the stochastic problem:

$$\sum_{t=1}^{\tau} (R_t - C_t) + (1 - \alpha)Y - (\beta + 1)\gamma \sum_{t=1}^{\min\{\tau, T\}} R_t \geq \theta, \quad \tau \geq 1. \tag{9}$$

Similarly, in project management, Long and Ohsato (2008) developed a deterministic schedule and used a project buffer for dealing with resource uncertainty. They determined the size of the buffer numerically; similarly, in Section 4.2, we determine the size of the cash buffer θ numerically as a function of problem data. We show that, as the uncertainty of revenues and costs increases, θ should increase to make the deterministic approximation solution feasible for the stochastic problem and provide a high-quality approximation for the stochastic problem.

These constraints imply that $B = \infty$, because the firm will never go bankrupt. In Section 4, we show, via computational experiments, that the optimal solution to the stochastic model in Problem (6) induces a low probability of firm bankruptcy over feasible Bolstr campaigns, which suggests that the constraints in (9) are unlikely to eliminate the optimal solution to the stochastic model; in Section 4.2, we evaluate the approximation quality of the deterministic model developed in this section for Problem (6), with very encouraging results.

Next, because the variables Y, M, and γ are continuous, we assume that the definition of T in Equation (1) holds exactly and deterministically:

$$\sum_{t=1}^{T} \gamma R_t = MY. \tag{10}$$

These simplifications result in a deterministic approximation to Problem (6) parameterized by the investment duration T:

$$\hat{z}_F(T) = \max_{Y, M, \gamma} \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1 + r_t)^t} - (\beta + 1)\gamma \sum_{t=1}^{T} \frac{R_t}{(1 + r_t)^t} + (1 - \alpha)Y$$

$$\text{s.t.} \sum_{t=1}^{T} \gamma R_t = MY \quad \text{(contractual obligation)}$$

$$\sum_{t=1}^{\tau} (R_t - C_t) + (1 - \alpha)Y$$

$$- (\beta + 1)\gamma \sum_{t=1}^{\min\{\tau, T\}} R_t \geq \theta, \ \tau \geq 1$$

$$\text{(cashflow constraints)}$$

$$\sum_{t=1}^{T} \frac{\gamma}{Y} \frac{R_t}{(1 + \delta_t)^t} \geq \max_{1 \leq i \leq n} \left\{ \frac{A_i}{y_i} \right\} + 1$$

$$\text{(investor participation)}$$

$$Y, \gamma \geq 0, M \geq 1.$$

$$\tag{11}$$

3.1. Analysis for Fixed $T \in \mathbb{N}$

In this subsection, we solve Model (11) for a fixed $T \in \mathbb{N}$. To begin our analysis, we point out some simplifications. First, we let $\hat{A} = \max_{1 \leq i \leq n} \left\{ \frac{A_i}{y_i} \right\}$ to simplify the exposition. Second, the contractual obligation constraint can be used to solve for $\gamma = \frac{MY}{\sum_{t=1}^{T} R_t}$, and γ is eliminated as a variable. In the subsequent analysis, it will be convenient to define the set.

$$X \triangleq \left\{ \tau \in \mathbb{N} : \sum_{t=1}^{\tau} (R_t - C_t) < \theta \right\}, \tag{12}$$

which indexes all of the time periods where the firm, without any investment, has cash flow below θ. It is also convenient to define the parameters $Z_\tau(T)$, which depend only on firm problem data and T:

$$Z_\tau(T) \triangleq (\hat{A} + 1)(\beta + 1) \frac{\sum_{t=1}^{\min\{\tau, T\}} R_t}{\sum_{t=1}^{T} \frac{R_t}{(1 + \delta_t)^t}} - (1 - \alpha). \tag{13}$$

The resulting model has the following solution.

Proposition 1. *Problem* (11), *with* $T \in \mathbb{N}$ *fixed and* $X \neq \emptyset$, *is feasible if and only if* $Z_\tau(T) < 0, \forall \tau \in X$ *and*

$$\max_{\tau \in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\} \le \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}, \text{ and it has}$$

the optimal solution $M^*(T) = \dfrac{(\hat{A}+1)\sum_{t=1}^{T}R_t}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}}$ *and* $\gamma^*(T) = \dfrac{(\hat{A}+1)Y^*(T)}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}}$,

where
- *if* $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \ge 0$, *then* $Y^*(T) =$

$\max_{\tau \in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$; *alternatively, if* $X = \emptyset$, *then* $Y^* = 0$,

$\gamma^* = 0$, *and* $M^* = 1$ *for all* $T \in \mathbb{N}$.

- *if* $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$, *then* $Y^*(T) =$

$\min_{\substack{\tau \notin X \\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$; *alternatively, if* $X = \emptyset$, *then*

$Y^*(T) = \min_{\substack{\tau \in \mathbb{N} \\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$ *for all* $T \in \mathbb{N}$.
The maximized NPV is $\hat{z}_F(T) = \sum_{t=1}^{\infty}\frac{R_t-C_t}{(1+r_t)^t} -$
$\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha)\right)Y^*(T)$.

Note that Proposition 1 only requires that the cash flows R_t and C_t be deterministic but does not require them to be linear. The feasibility constraints in Proposition 1 can be interpreted as financial conditions, where the firm can eventually survive on its own; as an example, where this is not possible, consider the case where $C_t > R_t$ for all t. In particular, $Z_\tau(T) < 0, \forall \tau \in X$ ensures that the firm is cash flow positive for all periods $\tau \in X$ because of the investment $Y^*(T)$, and the condition $Y^*(T) \le \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$ ensures that the firm can afford to pay back $M^*(T)Y^*(T)$ to investors and $\beta M^*(T)Y^*(T)$ to the platform.

If the firm's discount rates r_t are not too large relative to the investors' discount rates δ_t and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \ge 0$ holds, then we see that, intuitively, if $X = \emptyset$, then the firm does not need any investment, and $Y^*(T) = 0$. Alternatively, if $X \ne \emptyset$ and the feasibility conditions are satisfied, then $\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha)\right)Y^*(T)$ can be interpreted as the firm's cost for avoiding bankruptcy.

Alternatively, if $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$, then the firm's discount rates r_t are relatively larger than the investors' discount rates δ_t. Therefore, the firm's gain from the $(1-\alpha)Y$ investment at time 0 is greater than the firm's NPV of future payments to the investors and the platform. As a result, the firm benefits by raising a larger investment.

In Figure 1, we plot the average of the objective function value $\hat{z}_F(T)$ and the average of the optimal variables $(Y^*(T), M^*(T), \gamma^*(T))$ from Proposition 1 as a function of T over feasible Bolstr campaigns for the following parameter set. For each campaign, we let linear revenues $R_t = E[R_t]$ and costs $C_t = E[C_t]$, where $E[R_t]$ and $E[C_t]$ are given in Equation (7), and only use the given campaign's data, $\theta = 0$, $\alpha = 0.05$ and $\beta = 0.01$ (per a Bolstr memorandum), $\hat{A} = 0.1$ (i.e., a 10% NPV return for investors), and $r_t = \delta_t = 0.01$, $\forall t$ (the discount rate per period, typically a month). These results are useful in the sequel for interpreting the results for the stochastic problem where the level of variability in costs and revenues is small.

The plot in the upper left of Figure 1 suggests that $\hat{z}_F(T)$ converges rather quickly to an asymptote. In the next section, for linear cash flows, we derive conditions for which $\max_T \hat{z}_F(T)$ is attained when $T \to \infty$. However, our numerical results suggest that relatively small values of T, say $T \in \{18, \dots, 30\}$, suffice to attain almost all of the potential value of $\max_T \hat{z}_F(T)$. In the upper right plot of Figure 1, we observe that $Y^*(T)$ decreases rather quickly to an asymptote as well. In the lower left plot of Figure 1, we see that $M^*(T)$ is increasing in T, and in the lower right plot of Figure 1, we see that $\gamma^*(T)$ is decreasing in T, which intuitively aligns with the increased investment duration T. We point out that our model was only feasible for $T \ge 12$ for all feasible campaigns for $\theta = 0$; in the next section, for linear cash flows, we provide a rigorous proof that Problem (11) can only be feasible for large-enough T.

3.2. The Optimal Investment Horizon T^*
In the previous section, we fixed $T \in \mathbb{N}$ and solved Problem (11); we provided closed form expressions in Proposition 1 for $\gamma^*(T)$, $M^*(T)$, and $Y^*(T)$ for feasible problems. In this section, we find the optimal investment horizon T^*. Proposition 1 indicates that Model (11) becomes

$$\hat{z}_F(T) = \max_{T \in \mathbb{N}} \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1+r_t)^t}$$
$$- \left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha)\right)Y^*(T)$$
$$\text{s.t. } Z_\tau(T) < 0, \quad \tau \in X$$
$$\max_{\tau \in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$$
$$\le \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\},$$

(14)

where $Y^*(T)$ is defined in Proposition 1. It is convenient to define the term $R^\infty \triangleq \sum_{t=1}^{\infty}\frac{R_t}{(1+\delta_t)^t}$, which we assume is finite, to present the next set of results, which are valid for large T and characterize the structure of Problem (14).

Figure 1. (Color online) Illustration of Proposition 1 for Feasible Bolstr Campaigns with Linear Costs and Revenues
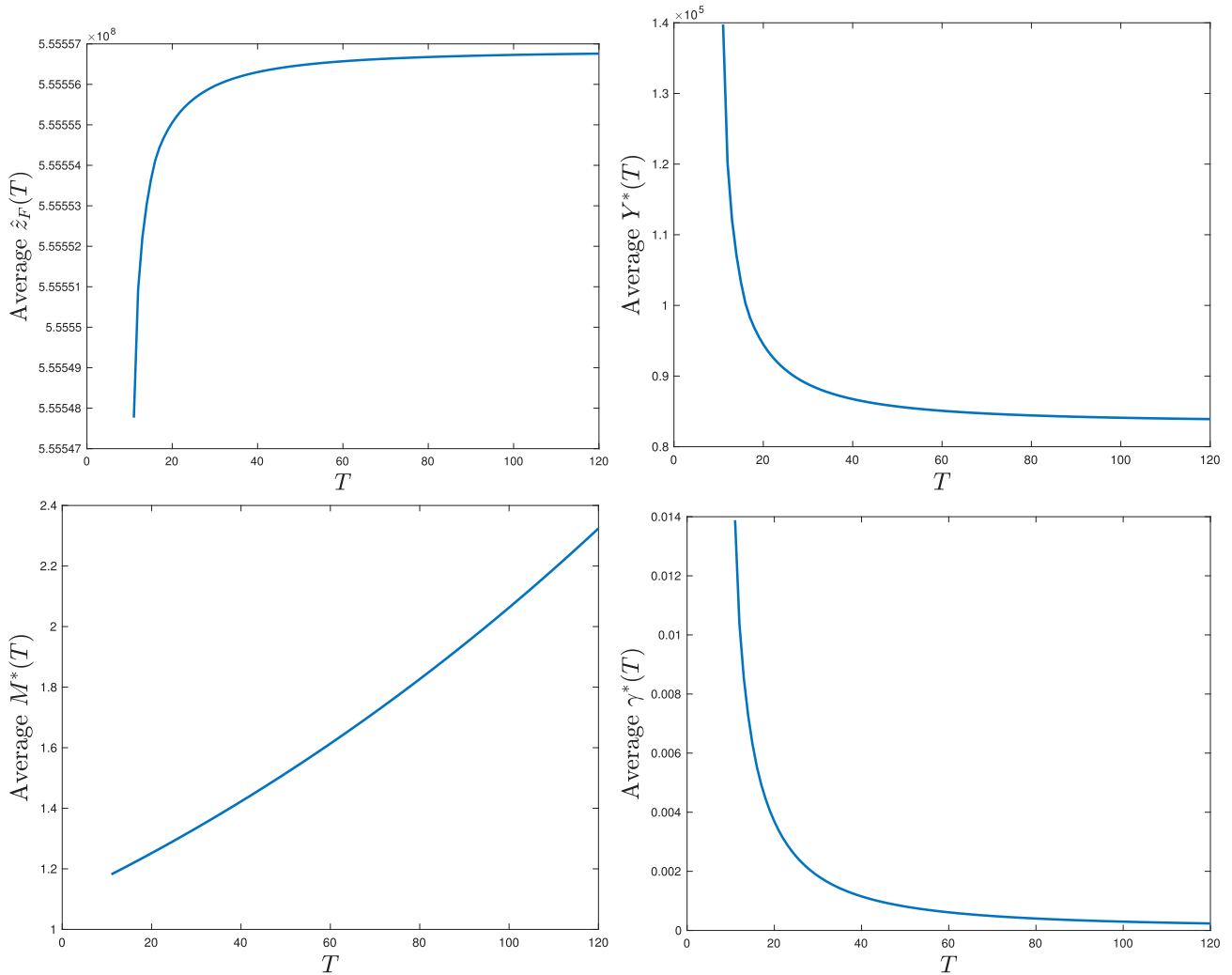
Lemma 1. If $(\hat{A} + 1)(\beta + 1)\sum_{t=1}^{\tau} R_t/R^{\infty} < (1-\alpha)$ for all $\tau \in X$, then $Z_\tau(T) < 0$ for all $\tau \in X$.

Lemma 2. For $\tau \in X$ and $Z_\tau(T) < 0$, $\max_{\tau \in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\}$ is strictly decreasing in T.

Lemma 3. For $\tau \notin X$ and $Z_\tau(T) > 0$, $\min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\}$ is strictly increasing in T.

The next lemma builds on Lemmas 2 and 3 to characterize the objective function of Problem (14).

Lemma 4. If $r_t = r \geq \delta = \delta_t, \forall t$ or $r_t = \delta_t, \forall t$, then the objective function of Problem (14) is increasing in T for large T.

Lemmas 2 and 3 suggest that $\max_{\tau \in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\} \leq \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\}$ is attained asymptotically. However,

this is not always possible. Consider the following example.

Example 1. Let the revenues $R_t = 10$ for $t \geq 1$; the costs are $C_1 = 15$, $C_2 = 5$, and $C_t = 10$ for $t \geq 3$, and $\theta = 0$. The set $X = \{1\}$, and for feasible values of $(\hat{A}, \alpha, \beta, \delta_t)$, $\max_{\tau \in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\} > 0$ for all feasible T. In contrast, $\sum_{t=1}^{\tau}(R_t - C_t) = 0$ for all $\tau \notin X$, implying that $\min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\} = 0$.

In the next subsection, we see that an assumption of linear cash flows resolves the issue encountered in the above counterexample, and Problem (14) is solvable analytically. Recall that the linearity of revenues and costs is supported by real data from Bolstr as explained in Section 2.1.

Figure 2. (Color online) The Form of the Function $f(\tau)$
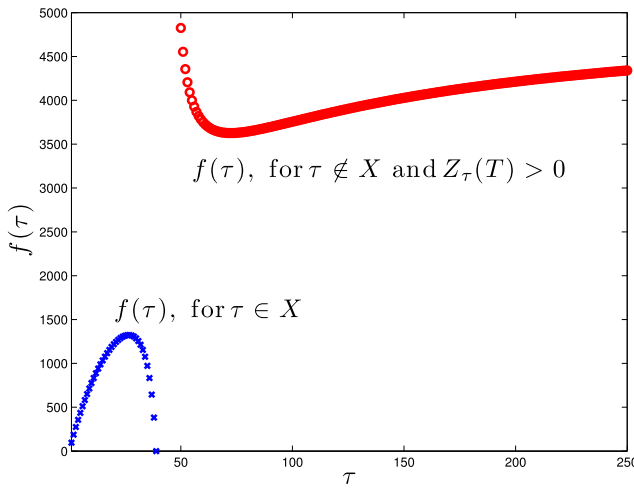
Linear Firm Costs and Revenues. Next, we consider Problem (14) when the costs and revenues are linear: $R_t = a + bt$ and $C_t = c + dt$. We could, for instance, parameterize (a, b, c, d) using the regression analysis on Bolstr data. The set X, for linear costs and revenues, simplifies to

$$X = \left\{ \tau \in \mathbb{N} : (a - c)\tau + (b - d)\frac{\tau(\tau + 1)}{2} < \theta \right\}. \quad (15)$$

The following function is useful for the subsequent analysis and results:

$$f(\tau) \triangleq \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} = \frac{(a - c)\tau + (b - d)\frac{\tau(\tau+1)}{2} - \theta}{\eta(T)\left(a\tau + b\frac{\tau(\tau+1)}{2}\right) - (1 - \alpha)}, \quad (16)$$

where $\eta(T) \triangleq \frac{(\hat{A}+1)(\beta+1)}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} > 0$. This function allows us to write $\max_{\tau \in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\} = \max_{\tau \in X} f(\tau)$ and $\min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\} = \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} f(\tau)$, where $\max_{\tau \in X} f(\tau)$ is a lower bound on $Y^*(T)$ to allow cash flow positivity at or above θ and $\min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} f(\tau)$ is an upper bound on $Y^*(T)$ to ensure that the firm can afford to pay back $M^*(T)Y^*(T)$ to investors and $\beta M^*(T)Y^*(T)$ to the platform (Figure 2). Our next two results characterize the cases where the set X is unbounded or empty, which results in infeasible and trivial firm optimization problems, respectively.

Lemma 5. *If (1) $b < d$ or (2) $b = d$ and $a < c + \theta$, then Problem (14) is infeasible.*

Under the conditions of Lemma 5, the firm will eventually go bankrupt for all values of (Y, M, γ).

Lemma 6. *If (1) $b > d$, $(b - d) \geq (c - a) + \theta$, and $(\beta + 1)$ $(\hat{A} + 1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1 - \alpha) \geq 0$ or (2) $b = d$ and $a \geq c + \theta$, then $Y^*(T) = 0$ for all $T \in \mathbb{N}$ in Problem (11).*

Under either of the conditions of Lemma 6, the firm does not need any outside funding and is cash flow positive at or above θ for all periods $\tau \geq 1$.

Another set of conditions where the firm is cash flow positive at or above θ for all periods $\tau \geq 1$ is $b > d$, $b - d \geq (c - a) + \theta$, and $(\beta + 1)(\hat{A} + 1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1 - \alpha) < 0$. In the next lemma, we show that, under these conditions, if feasible, the firm borrows money from investors, although it is already cash flow positive at or above θ. The reason is that, under these conditions, the firm's gain from raising an investment at time $t = 0$ is larger than the NPV of its future monthly payments to the investors and the platform because of its high discount rates.

Lemma 7. *If $b > d$, $b - d \geq (c - a) + \theta$, and $(\beta + 1)(\hat{A} + 1)$ $\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1 - \alpha) < 0$, then for $\theta \leq \theta_L(T)$, where $\theta_L(T)$ is a function of T and problem parameters, we have for all $T \in \mathbb{N}$:*

if $cb - da \geq 0$, then $Y^*(T) = \min\{f(\lfloor \tau^{**} \rfloor), f(\lceil \tau^{**} \rceil)\}$, where τ^{**} is real and equal to

$$\begin{aligned} \tau^{**} = &[(1 - \alpha)(b - d) - \eta(T)\theta b \\ &+ [((1 - \alpha)(b - d) - \eta(T)\theta b)^2 \\ &- \eta(T)(cb - da)((2(c - a) - (b - d))(1 - \alpha) \\ &+ \eta(T)\theta(2a + b))]^{1/2}]/[(cb - da)\eta(T)]. \end{aligned}$$

if $cb - da < 0$, then $Y^*(T) = (b - d)/\eta(T)b$.

We have characterized all combinations of (a, b, c, d), except the case where $b > d$ and $b - d < (c - a) + \theta$. We break this case into two subcases: (1) $b > d$, $b - d < (c - a) + \theta$, and $cb > da$, and (2) $b > d$, $b - d < (c - a) + \theta$, and $cb \leq da$. These cases result in Problem (14) being feasible with a nontrivial solution. In particular, Case 1 includes firms that have cash flow shortages for a limited time but expectations of positive cash flows in the future. Case 2 includes firms that have positive cash flows, but they are below θ, and the firm receives investment to increase its cash flow to at least θ. Recall that θ is a cash buffer in the deterministic approximation problem to account for cash flow uncertainties in the stochastic problem. We begin by characterizing the set X for both of these cases.

Lemma 8. *If $b > d$ and $b - d < (c - a) + \theta$, then $X = \left\{ \tau \in \mathbb{N} : 1 \leq \tau < \frac{\left(c - a - \frac{b-d}{2}\right) + \sqrt{\left(c - a - \frac{b-d}{2}\right)^2 + 2\theta(b - d)}}{b - d} \right\}$.*

From the set X, it is clear that, as θ increases, the number of periods where the firm has a cash flow below θ increases. The following lemma indicates that the problem is not feasible for θ larger than a threshold.

Lemma 9. *The first constraint in Problem* (14), $Z_\tau(T) < 0$ *for all* $\tau \in X$, *is equivalent to the conditions* $\theta \le \hat{\theta}(T)$ *and* $(ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \ge 0$, *where* $\hat{\theta}(T)$ *is a function of T and problem parameters.*

Note that the left-hand side of the second condition in Lemma 9, $(ad - bc) + (b - d)$ $\cdot \sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \ge 0$, is increasing in T, albeit asymptotically, because of $\eta(T)$. Therefore, if it is possible for the first constraint of Problem (14) to hold, it will be feasible for a large-enough T and a small-enough θ. We next analyze Cases 1 and 2, building on the condition in Lemma 9, to address the second constraint in Problem (14).

Case 1. $b > d$, $b - d < (c - a) + \theta$, and $cb > da$. The next three lemmas characterize the second constraint in Problem (14) for Case 1: $b > d$, $b - d < (c - a) + \theta$, and $cb > da$, assuming that the feasibility conditions of the first constraint in Lemma 9 hold.

Lemma 10. *If* $b > d$, $cb > da$, *and* $(ad - bc) + (b - d)$ $\cdot \sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \ge 0$, *then for* $\theta \le \tilde{\theta}(T)$ *and* $b - d < (c - a) + \theta$, *where* $\tilde{\theta}(T)$ *is a function of T and problem parameters:*
If $\theta \ge \frac{((b-d)-2(c-a))(1-\alpha)}{\eta(T)(2a+b)}$, *then* $\max_{\tau \in X} f(\tau) = \max\{f(\lfloor \tau^* \rfloor),$ $f(\lceil \tau^* \rceil)\}$, *where* τ^* *is real and equal to*

$$\tau^* = [(1 - \alpha)(b - d) - \eta(T)\theta b$$
$$- [((1 - \alpha)(b - d) - \eta(T)\theta b)^2$$
$$- \eta(T)(cb - da)((2(c - a) - (b - d))(1 - \alpha)$$
$$+ \eta(T)\theta(2a + b))]^{1/2}]/[(cb - da)\eta(T)].$$

Otherwise, $\max_{\tau \in X} f(\tau) = \frac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}$.

Lemma 11. *If* $b > d$, $cb > da$, *and* $(ad - bc) + (b - d)$ $\cdot \sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \ge 0$, *then for* $\theta \le \tilde{\theta}(T)$ *and* $b - d < (c - a) + \theta$, *where* $\tilde{\theta}(T)$ *is a function of T and problem parameters, we have* $\min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \{f(\tau)\} = \min\{f(\lfloor \tau^{**} \rfloor),$ $f(\lceil \tau^{**} \rceil)\}$, *where* τ^{**} *is real and equal to*

$$\tau^{**} = [(1 - \alpha)(b - d) - \eta(T)\theta b$$
$$+ [((1 - \alpha)(b - d) - \eta(T)\theta b)^2$$
$$- \eta(T)(cb - da)((2(c - a) - (b - d))(1 - \alpha)$$
$$+ \eta(T)\theta(2a + b))]^{1/2}]/[(cb - da)\eta(T)].$$

Lemma 12. *If* $b > d$, $cb > da$, *and* $(ad - bc) + (b - d)$ $\cdot \sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \ge 0$, *then* $\exists \bar{\theta}(T)$, *such that, for* $\theta \le \bar{\theta}(T)$ *and* $b - d < (c - a) + \theta$, *where* $\bar{\theta}(T)$ *is a function of T and problem parameters, the inequality* $\max_{\tau \in X}\{f(\tau)\} \le \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \{f(\tau)\}$ *holds for T large enough.*

Lemmas 10–12 prove that, for linear cash flows and Case 1, if θ is small enough and T is large enough, Problem (14) is feasible. We next accomplish the same task for Case 2.

Case 2. $b > d$, $b - d < (c - a) + \theta$, and $cb \le da$. The next three lemmas characterize the second constraint in Problem (14) for Case 2: $b > d$, $b - d < (c - a) + \theta$, and $cb \le da$, assuming that the feasibility conditions of the first constraint in Lemma 9 hold.

Lemma 13. *If* $b > d$, $cb \le da$, *and* $(ad - bc) + (b - d)$ $\cdot \sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \ge 0$, *then for* $\theta \le \tilde{\theta}(T)$ *and* $b - d < (c - a) + \theta$, *where* $\tilde{\theta}(T)$ *is a function of T and problem parameters, we have* $\max_{\tau \in X} f(\tau) = \frac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}$.

Lemma 14. *If* $b > d$, $cb \le da$, *and* $(ad - bc) + (b - d)$ $\cdot \sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \ge 0$, *then for* $\theta \le \tilde{\theta}(T)$ *and* $b - d < (c - a) + \theta$, *where* $\tilde{\theta}(T)$ *is a function of T and problem parameters, we have* $\min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \{f(\tau)\} = (b-d)/\eta(T)b$.

Note that, from $b > d$ and $cb \le da$, we conclude that $a \ge c$. Conditions $b > d$ and $a \ge c$ represent a firm that has higher revenues than costs in all periods, with the revenue growth larger than that of cost; however, the firm's cash flow is not above θ in all periods, which drives the need for investment.

Lemma 15. *If* $b > d$, $cb \le da$, *and* $(ad - bc) + (b - d)$ $\cdot \sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \ge 0$, *then* $\exists \bar{\theta}(T)$, *such that, for* $\theta \le \bar{\theta}(T)$ *and* $b - d < (c - a) + \theta$, *where* $\bar{\theta}(T)$ *is a function of T and problem parameters, the inequality* $\max_{\tau \in X}\{f(\tau)\} \le \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \{f(\tau)\}$ *holds.*

The conclusion of Lemmas 12 and 15, $\max_{\tau \in X}\{f(\tau)\} \le \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \{f(\tau)\}$, guarantees that the firm is cash flow positive at or above θ in all months for Cases 1 and 2, respectively, which is only possible if θ is small enough. Lemmas 2 and 3 prove that the inequality $\max_{\tau \in X}\{f(\tau)\} \le \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \{f(\tau)\}$, if feasible, is feasible for T large enough. Thus, Problem (14), under linear cash flows and Cases 1 and 2, is feasible for T large enough and θ small enough. The results in Lemmas 7, 10, and 13 provide closed form solutions for $Y^*(T)$ under all nontrivial cases, and Proposition 1 provides closed form solutions for $M^*(T)$ and $\gamma^*(T)$ as a function of $Y^*(T)$. We found that $T = 120$ worked well to generate high-quality approximations for problems parameterized by real Bolstr data. These optimal variables for Problem (14) are used as approximate solutions for the stochastic model in Problem (6), with quality that we explore in the next section.

To complete the analysis of the deterministic problem, we now collect all of these results to solve Problem (14) for the cases where $r_t = r \ge \delta = \delta_t, \forall t$ or $r_t = \delta_t, \forall t$.

Proposition 1 and Lemmas 7, 10, and 13 for linear cash flows provide closed form solutions for the optimal $\gamma^*(T)$, $M^*(T)$, and $Y^*(T)$, assuming model feasibility and a fixed $T \in \mathbb{N}$. Lemma 4 indicates that the objective of Problem (14) is strictly increasing in T. Lemmas 1–3 prove that Problem (14), if feasible, is feasible for large-enough T. We also note that $\lim_{T\to\infty}\eta(T) = \lim_{T\to\infty}\frac{(\hat{A}+1)(\beta+1)}{\sum_{t=1}^{T}\frac{(a+bt)}{(1+\delta_t)^t}} = (\hat{A}+1)(\beta+1)/R^\infty$. Lemmas 11, 12, 14, and 15 prove, for Cases 1 and 2, respectively, that, for linear cash flows and large-enough T, the model is feasible. Together, these results solve Problem (14) for linear costs and revenues, which we summarize in the next propositions. Specifically, in Proposition 2, we characterize the optimal solutions for $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$, and in Proposition 3, we do the same for $(\beta+1)\cdot(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$.

Proposition 2. *For* $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$ *and either* $r_t = r \geq \delta = \delta_t, \forall t$ *or* $r_t = \delta_t, \forall t$, *if* $b > d$, $b - d < (c-a) + \theta$, $\theta \leq \lim_{T\to\infty}\bar{\theta}(T)$, *and* $(ad-bc) + (b-d)\sqrt{(a+b/2)^2 + 2(1-\alpha)b\frac{R^\infty}{(\hat{A}+1)(\beta+1)}} \geq 0$, *then*

if $cb > da$ *and* $\theta \geq \frac{((b-d)-2(c-a))(1-\alpha)(\hat{A}+1)(\beta+1)}{R^\infty(2a+b)}$, *then* $T^* = \infty$, $M^* = \infty$, *and* $Y^* = \max\{f(\lfloor\tau^*\rfloor), f(\lceil\tau^*\rceil)\}, \gamma^* = \frac{(\hat{A}+1)Y^*}{R^\infty}$, *where*

$$\tau^* = [(1-\alpha)(b-d) - \theta b(\hat{A}+1)(\beta+1)/R^\infty]$$

$$/[(cb-da)(\hat{A}+1)(\beta+1)/R^\infty]$$

$$- [[((1-\alpha)(b-d) - \theta b(\hat{A}+1)(\beta+1)/R^\infty)^2$$

$$- (\hat{A}+1)(\beta+1)/R^\infty(cb-da)((2(c-a)$$

$$- (b-d))(1-\alpha) + (\hat{A}+1)(\beta+1)$$

$$/R^\infty\theta(2a+b))]^{1/2}]/[(cb-da)(\hat{A}+1)(\beta+1)/R^\infty].$$

Otherwise, $T^* = \infty$, $M^* = \infty$, $Y^* = \frac{a+b-c-d-\theta}{(\hat{A}+1)(\beta+1)(a+b)/R^\infty - (1-\alpha)}$, *and* $\gamma^* = \frac{(\hat{A}+1)\frac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}}{R^\infty}$.

The firm's maximized NPV is $\hat{z}_F = \sum_{t=1}^{\infty}\frac{a-c+(b-d)t}{(1+r_t)^t} - \left((\beta+1)\cdot(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha)\right)Y^*$.

Proposition 3. *For* $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$ *and* $r_t = r \geq \delta = \delta_t, \forall t$, *if* $b > d$, $b - d \geq (c-a) + \theta$, *and* $\theta \leq \lim_{T\to\infty}\theta_L(T)$ *or if* $b > d$, $b - d < (c-a) + \theta$, $\theta \leq \lim_{T\to\infty}\bar{\theta}(T)$, *and* $(ad-bc) + (b-d)\sqrt{(a+b/2)^2 + 2(1-\alpha)b\frac{R^\infty}{(\hat{A}+1)(\beta+1)}} \geq 0$, *then*

if $cb - da \geq 0$, *then* $Y^*(T) = \min\{f(\lfloor\tau^{**}\rfloor), f(\lceil\tau^{**}\rceil)\}$,
if $cb - da < 0$, *then* $Y^*(T) = (b-d)/\eta(T)b$,
where τ^{**} *is real and equal to*

$$\tau^{**} = [(1-\alpha)(b-d) - \theta b(\hat{A}+1)(\beta+1)/R^\infty]$$

$$/[(cb-da)(\hat{A}+1)(\beta+1)/R^\infty]$$

$$+ [[((1-\alpha)(b-d) - \theta b(\hat{A}+1)(\beta+1)/R^\infty)^2$$

$$- (\hat{A}+1)(\beta+1)/R^\infty(cb-da)((2(c-a)$$

$$- (b-d))(1-\alpha) + (\hat{A}+1)(\beta+1)$$

$$/R^\infty\theta(2a+b))]^{1/2}]/[(cb-da)(\hat{A}+1)(\beta+1)/R^\infty],$$

and $T^* = \infty$, $M^* = \infty$, $\gamma^* = \frac{(\hat{A}+1)Y^*}{R^\infty}$. *The firm's maximized NPV is* $\hat{z}_F = \sum_{t=1}^{\infty}\frac{a-c+(b-d)t}{(1+r_t)^t} - \left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha)\right)Y^*$.

From Propositions 2 and 3, we see that, if $r_t = r \geq \delta = \delta_t, \forall t$ or $r_t = \delta_t, \forall t$, then Y^* and γ^* are finite, but M^* and T^* are infinite. This solution corresponds to a financial perpetuity (an annuity with no termination) with nonfixed payments that are a function of firm revenues. Note that perpetuities are common in modern business (e.g., anyone can purchase a perpetuity through Bank of America's Merrill Edge brokerage). More prominent examples include LeBron James, a four-time NBA MVP (National Basketball Association Most Valuable Player), and soccer star David Beckham having lifetime contracts with Nike and Adidas, respectively (Novy-Williams 2015). Furthermore, Gerber et al. (2012) provide a study of three crowdfunding platforms and showed that many individuals and firms use crowdfunding to make direct and long-term connections with investors; a perpetuity precisely achieves a long-term connection.

Our perpetuity solution can also be interpreted as a pseudotype of equity crowdfunding. Ross et al. (2002) explain that the present value of stock is equivalent to the discounted present value of all future dividends, which are typically a share of profits (O'Sullivan and Sheffrin 2007, p. 198). If we replace profits with revenues, we obtain our perpetuity contract. In Section 5, we formally study the difference between our model and an equity crowdfunding model that shares profits rather than revenues, and we show that the revenue-sharing contract is superior.

4. Analysis of Stochastic Model
4.1. Simulation-Based Numerical Optimization of Stochastic Model

To solve Problem (6), we let revenues R_t and costs C_t of each of the 56 Bolstr campaigns follow the random walk processes in (8). We set the highest allowable

values of σ^r and σ^c as $\mu^r/3$ and $\mu^c/3$, respectively, so that revenues and costs are nonnegative with high probability. More specifically, we consider $\sigma^r = \mu^r/k$ and $\sigma^c = \mu^c/k$ for $k \in \{3, 4, 5, 6, 7, 8, 9, 10, 15\}$ along with the additional deterministic case of $k \to \infty$.

In our base parameter set, we let $r_t = \delta_t = 0.01$ for all t, $\alpha = 0.05$, $\beta = 0.01$, and $\hat{A} = 0.1$ (i.e., a 10% NPV return for investors); note that r is the discount rate per period, which is typically a month, and that the choice of $\alpha = 0.05$ and $\beta = 0.01$ is supported by a Bolstr memorandum. Many online platforms, such as Bolstr, Lending Club, and Prosper, charge a 1% servicing fee for collecting and processing payments. These online lenders usually charge borrowers origination fees of typically $1\% - 8\%$.

We approximate an infinite horizon by considering $t \in \{1, \ldots, N\}$, where $N = 1,000$; we selected this value of N, so that $N > \max\{B, T\}$ holds with high probability. Furthermore, there is no evidence that revenue-sharing contracts at Bolstr and Localstake last longer than 1,000 months, and as pointed out earlier, most Bolstr campaigns last 2–5 years. For analyzing Problem (6) numerically, we discretized the (Y, M, γ) space. In particular, we considered values of $Y \in \{0, \Delta_Y, 2\Delta_Y, \ldots, \bar{Y}\}$, $M \in \{1, 1 + \Delta_M, 1 + 2\Delta_M, \ldots, \bar{M}\}$, and $\gamma \in \{0, \Delta_\gamma, 2\Delta_\gamma, \ldots, \bar{\gamma}\}$, where $(\Delta_Y, \Delta_M, \Delta_\gamma, \bar{Y}, \bar{M}, \bar{\gamma}) = (5,000, 0.25, 0.01, 2,000,000, 3, 1)$ were chosen to balance computational time and solution quality.

For each campaign, we used Monte Carlo simulation to generate $m = 1,000$ realizations of the $(R_1, \ldots, R_N)$ and $(C_1, \ldots, C_N)$ vectors, which allowed us to generate m realizations of the T and B random variables for each (Y, M, γ) tuple in the discretized set. Then, for each variable tuple, averaging over the m trials, we estimate $E\left(\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r)^t}\right)$, $E\left(\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r)^t}\right)$, and $E\left(\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+\delta)^t}\right)$,

which allowed us to evaluate the feasibility of the variable tuple. Finally, we evaluated the objective function for each feasible (Y, M, γ) tuple and chose the one that maximizes the objective function as the optimal solution. Depending on the standard deviations, 90%–95% of the 56 Bolstr campaigns were feasible.

4.2. Evaluation of Deterministic Approximation in Section 3

In this subsection, we evaluate the value of the approximate Problem (14) with respect to that of the stochastic Problem (6) for T large enough. For the numerical results presented below, we consider $T = 120$, although the performance is insensitive for larger T. In Section 4.3, we show that, if we use the approximation solutions in the true stochastic model, the expected investment horizon T is close to the choice of $T = 120$. The main parameters that we vary for each campaign are the standard deviations σ^r and σ^c; all other problem parameters are listed above. As mentioned previously, we consider $\sigma^r \in [0, \mu^r/3]$ and $\sigma^c \in [0, \mu^c/3]$, so that revenues and costs are nonnegative with high probability.

For each level of variability, we find the optimal $\theta \in [0, 2M]$, which makes the approximation solutions $(Y^*(\theta), M^*(\theta), \gamma^*(\theta))$ feasible for Problem (6) and results in the minimum approximation error for the stochastic problem. We denote this value of θ by θ^* in the sequel. In the left panel of Figure 3, we show that, as cash flow variability increases, θ^* increases to provide a larger cash buffer to absorb the additional variability.

The right panel of Figure 3 shows the percentage of feasible campaigns for Problem (6) that are also feasible for Problem (11) for θ^* and that can, therefore, be approximated via the approximation Problem (14). As

Figure 3. (Color online) (Left) The Average of θ^* over Feasible Campaigns; (Right) the Percentage of Feasible Campaigns for Problem (6) That Are Feasible for Problem (14)
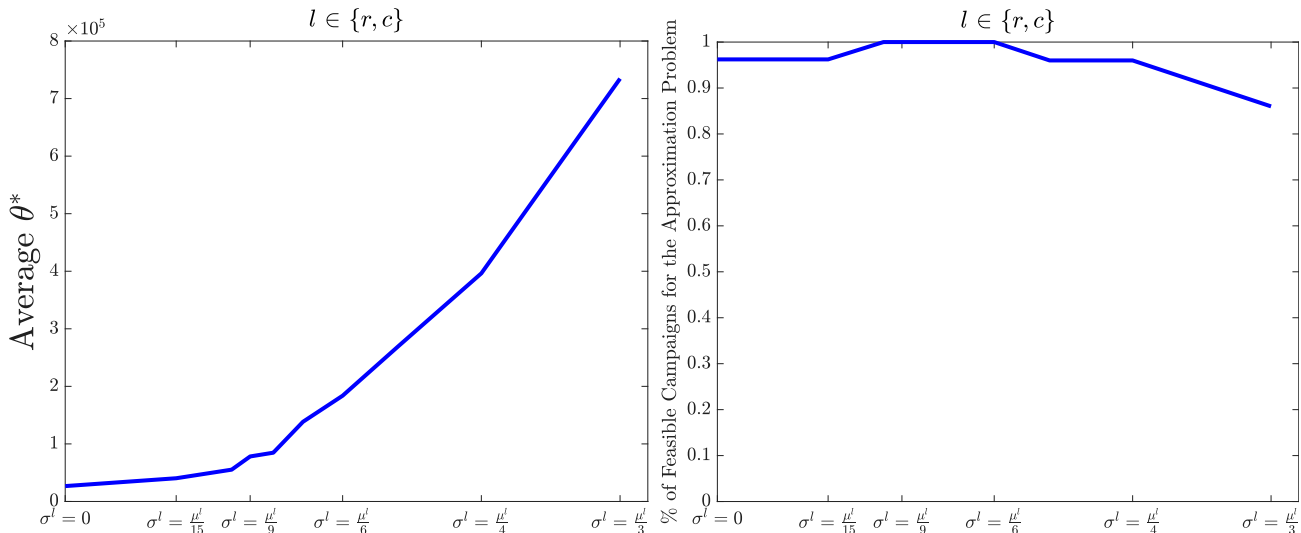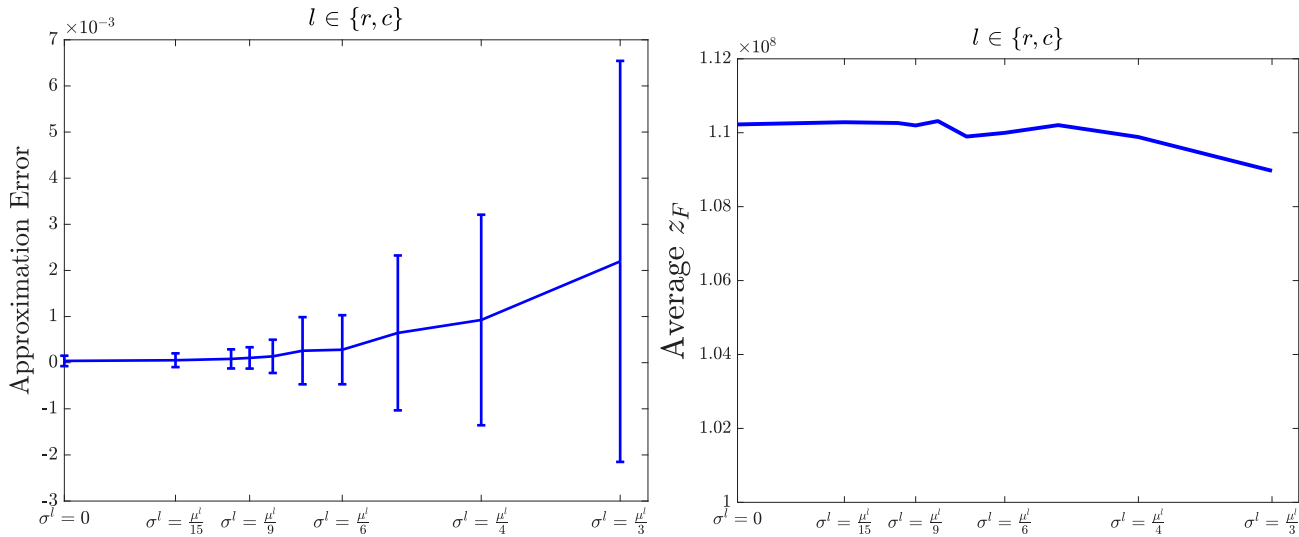
Figure 4. (Color online) (Left) Evaluation of the Approximation Quality of Problem (14) for Stochastic Problem (6); (Right) z_F over Feasible Campaigns

the level of cash flow uncertainty decreases, the approximation Problem (14) provides an approximate solution for Problem (6) for almost all campaigns.

For evaluating the quality of the approximation, we input the approximation solutions $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$ into Model (6) and compare the firm's expected NPV under this solution with the true optimal expected NPV z_F calculated numerically. In the left panel of Figure 4, we present the average error over all feasible Bolstr campaigns as a function of the standard deviations σ^r and σ^c. The length of each bar above and below the average value is equal to the standard deviation of the error over feasible campaigns. We observe that, for the highest level of variability, $\sigma^r = \mu^r/3$ and $\sigma^c = \mu^c/3$, the average error is 0.2% with a standard deviation of 0.4%. As variability decreases, the mean and standard deviation of the errors decrease, such that, for $k \to \infty$, the mean and standard deviation of the errors are 0.003% and 0.01%, respectively. For $k = 6$ ($\sigma^r = \mu^r/6$ and $\sigma^c = \mu^c/6$), the average error is within 0.03%. Consequently, we expect that the approximation quality is acceptable for reasonable levels of variability encountered in practice, and we conclude that the approximation problem provides tractable solutions for the intractable stochastic problem.

Finally, the average values of z_F as a function of cash flow variability are shown in the right panel of Figure 4. As variability decreases, z_F is increasing rather significantly for higher values of cash flow variability and very slowly for smaller values of variability. In other words, firms can increase their maximized expected NPV significantly by slightly decreasing their cash flow uncertainty in very uncertain environments. However, when cash flow uncertainty is not significant, the maximized expected NPV is slowly decreasing in the standard deviations σ^r and σ^c, and it is rather robust to their precise values.

4.3. Estimations of the *T* and *B* Distributions

In this subsection, we analyze the distributions of B, the firm's stochastic bankruptcy time, and T, the stochastic duration of the contract, for feasible campaigns under both the stochastic and approximation solutions. We present results for two qualitatively different campaigns that are feasible for all $k \geq 3$. One campaign, Campaign 1, has a relatively high bankruptcy probability for $k = 3$ (as representative of a risky firm), and the other campaign, Campaign 2, has a zero bankruptcy probability for $k = 3$ (as representative of a risk-less firm). The estimated probabilities $P(B < \infty)$ are given in Table 1 for the

Table 1. Estimated Probabilities $P(B < \infty)$

$k(\sigma^r = \mu^r/k$ and $\sigma^c = \mu^c/k)$	$P(B < \infty)$ for Campaign 1 for (Y^*, M^*, γ^*)	$P(B < \infty)$ for Campaign 2 for (Y^*, M^*, γ^*)	$P(B < \infty)$ for Campaign 1 for $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$	$P(B < \infty)$ for Campaign 2 for $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$
3	0.061	0	0.059	0
4	0.008	0	0.005	0
5	0.001	0	0.002	0
6	0.001	0	0.001	0
∞	0	0	0	0

Figure 5. (Color online) The Average of $P(B < \infty)$ over Feasible Campaigns
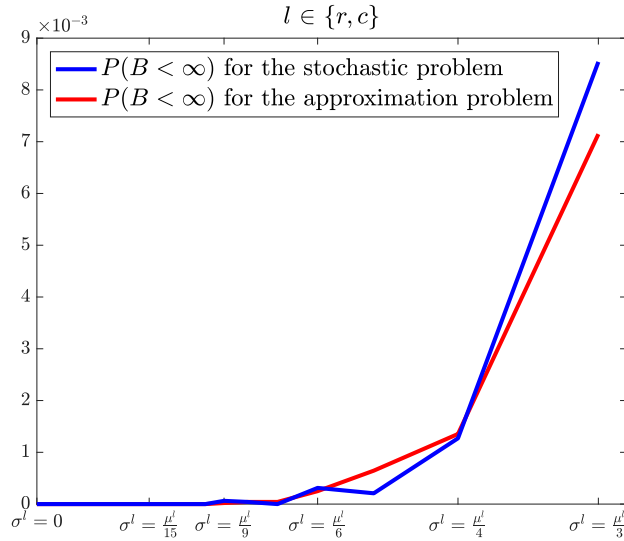
optimal and approximate solutions, respectively, as a function of k. We also evaluated the estimated probabilities $P(B < T)$, and they are identical to those in Table 1. We see that, as cash flow variability decreases, the firm's bankruptcy probability decreases for Campaign 1 (and Campaign 2's probability remains at zero) for both optimal and approximate variables; furthermore, for a given k, the probabilities are mostly identical across the two sets of variables, providing further evidence of the quality of the approximation.

Figure 5 shows the average of $P(B < \infty)$ over feasible Bolstr campaigns under both stochastic and approximation solutions. The approximation solutions result in almost the same bankruptcy probabilities as the

stochastic optimal solutions for feasible Bolstr campaigns over different levels of cash flow variability.

In the left plots of Figures 6 and 7, we provide the estimated distributions of T for $k \in \{3, 4, 5, \infty\}$ for Campaigns 1 and 2, respectively, for Problem (6) under the stochastic optimal solutions. Visually, we see that the mean and the standard deviation of T for the stochastic optimal solutions are decreasing in k (we confirmed this numerically). In other words, as k increases, the cash flow standard deviations σ^r and σ^c decrease, resulting in a problem with less uncertainty, which requires less investment Y^* and can be paid back earlier with more certainty about the investment duration. Furthermore, when $k \to \infty$, resulting in $\sigma^r = \sigma^c = 0$, our stochastic model returns a deterministic investment duration.

In the right plots of Figures 6 and 7, we see the estimated distributions of T for $k \in \{3, 4, 5, \infty\}$ for Campaigns 1 and 2, respectively, for Problem (6) under the approximation solutions. Similar to the left plots of Figures 6 and 7, the standard deviation of T is decreasing in k. However, the mean of the investment horizon is approximately the same for different levels of variability and close to $T = 120$, which we set earlier for the approximation problem. Therefore, if we use the approximation solution for setting the contract terms (Y, M, γ), for T large enough, then the expected investment horizon is close to T for all firms with different levels of variability.

Finally, because we observed that the firm can complete the repayment of the revenue-sharing contract within our horizon of $N = 1,000$ months, this value of N is unlikely a limiting factor of our numerical analysis. We also expect that it is unlikely that the firm will go bankrupt at some time $t > N$ under the less
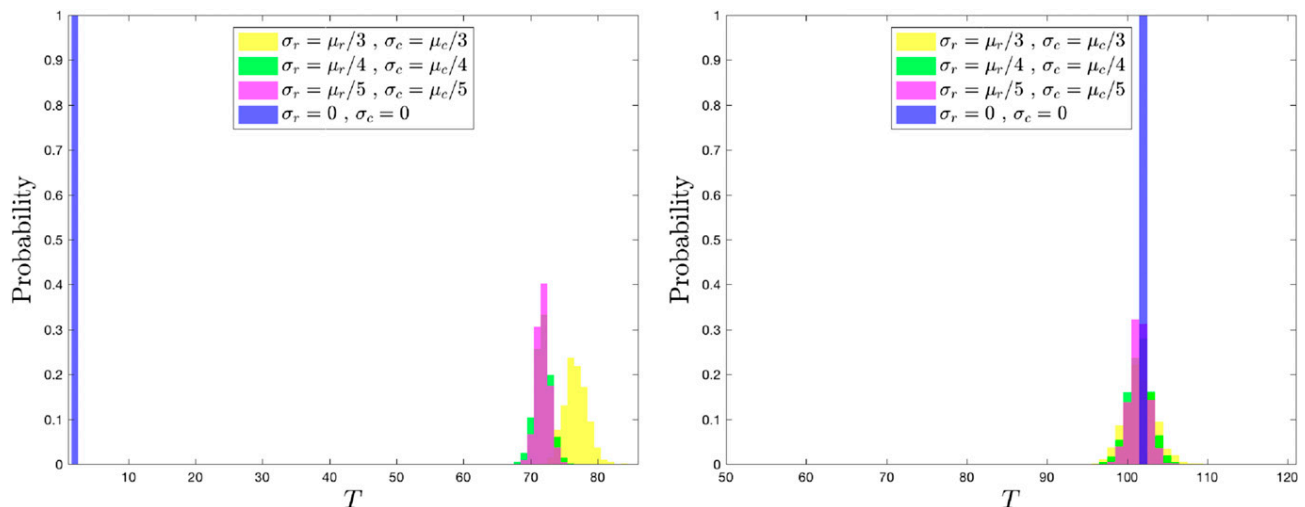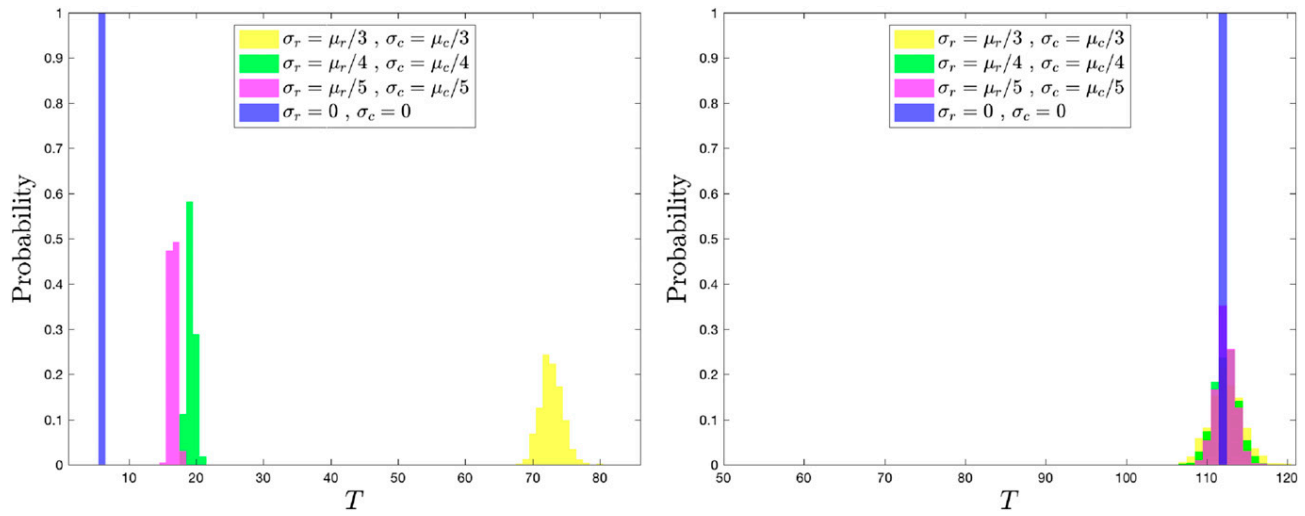
Figure 6. (Color online) Estimated Distributions of T in the Stochastic Problem for (Left) (Y^*, M^*, γ^*) and (Right) $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$, for Campaign 1

Figure 7. (Color online) Estimated Distributions of T in the Stochastic Problem for (Left) (Y^*, M^*, γ^*) and (Right) $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$, for Campaign 2


restrictive financial conditions because of the absence of repayment requirements, and we believe that our estimates of the probability of firm bankruptcy are not limited by the value of $N = 1,000$.

4.4. Sensitivity Analysis for Investors' Opportunity Cost

In this subsection, we perform sensitivity analysis for Problem (6) with respect to investors' opportunity cost $\hat{A}$. Figure 8 shows the average of z_F over the feasible Bolstr campaigns for different values of $\hat{A}$ as a function of k. In this subsection, we are analyzing the firm's problem for the cases where investors' opportunity cost is high, and specifically, we are analyzing the optimal solutions under a revenue-sharing contract with $\hat{A} \in \{0.1, 0.2, 0.3, 0.4\}$.

Figure 8. (Color online) The Average of z_F over Feasible Campaigns
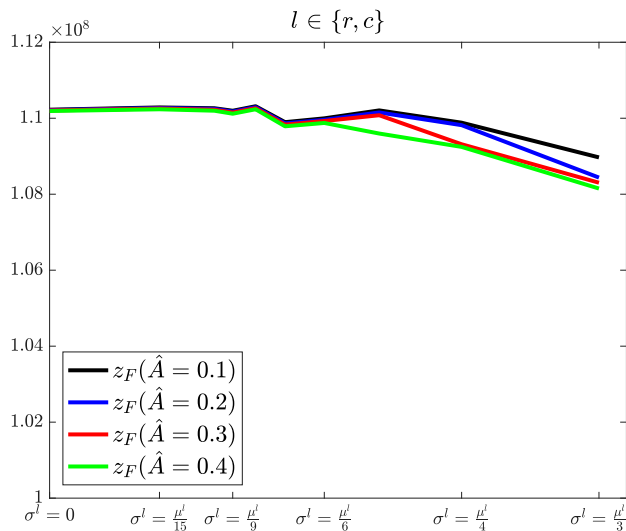
As shown in Figure 8, as investors' opportunity cost increases, the firm's maximized NPV decreases. However, the firm's maximized NPV is rather robust with respect to $\hat{A}$ under a flexible revenue-sharing contract. Therefore, firms can benefit from flexible revenue-sharing contracts even with investors with moderately high opportunity costs, and the benefit increases as cash flow variability decreases.

5. Equity Crowdfunding

In this section, we analyze an equity crowdfunding investment and compare it with our revenue-sharing contract. According to Bradford (2012), pp. 33–34, investors purchase equity from the firm and receive returns in the form of profit sharing. Similarly, Ross et al. (2002) explain that the present value of equity is equal to the discounted present value of all future dividends, which are typically shares of profit (O'Sullivan and Sheffrin 2007).

As in the analyses of previous sections, we assume that the firm raises Y from n investors, such that $Y = \sum_{i=1}^{n} y_i$. Investors, in exchange for their equity investments, collectively receive η percent of the firm's monthly profit indefinitely divided among the investors proportional to their individual investment amounts.

As in our previous analyses, the investors' discount rate in period t is δ_t, and investor i's expected NPV is given by $E\left[\sum_{t=1}^{B} \frac{y_i}{Y} \frac{\eta \max\{R_t - C_t, 0\}}{(1 + \delta_t)^t}\right]$, where $\max\{R_t - C_t, 0\}$ captures profit sharing. Thus, the investors' participation constraints are $E\left[\sum_{t=1}^{B} \frac{y_i}{Y} \frac{\eta \max\{R_t - C_t, 0\}}{(1 + \delta_t)^t}\right] - y_i \geq A_i$, $i = 1, \ldots, n \Leftrightarrow E\left[\sum_{t=1}^{B} \frac{\eta}{Y} \frac{\max\{R_t - C_t, 0\}}{(1 + \delta_t)^t}\right] \geq \hat{A} + 1$. As in our previous analyses, we assume that the platform charges an origination fee α and a servicing fee β. The firm's discount rate in period t is again r_t. The expected NPV

of the firm after the crowdfunding investment (from the firm's perspective) is $V \triangleq E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t}\right] + (1-\alpha)Y - (\beta+1)\eta E\left[\sum_{t=1}^{B} \frac{\max\{R_t - C_t, 0\}}{(1+r_t)^t}\right]$. As a result of their investment, investors collectively own a $\frac{Y}{V}$ proportion of the company's value, and the remaining $\left(1 - \frac{Y}{V}\right)$ proportion belongs to the firm. The expected NPV of the firm's remaining ownership is given by $\left(1 - \frac{Y}{V}\right)V$, which results in the firm's maximization problem being defined as

$$z_E = \max_{Y,\eta} E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t}\right] - \alpha Y$$

$$- (\beta+1)\eta E\left[\sum_{t=1}^{B} \frac{\max\{R_t - C_t, 0\}}{(1+r_t)^t}\right]$$

$$\text{s.t. } B = \min\left\{\hat{B} \geq 1 : \sum_{t=1}^{\hat{B}} (R_t - C_t) + (1-\alpha)Y\right.$$

$$\left. - (\beta+1)\eta \sum_{t=1}^{\hat{B}} \max\{R_t - C_t, 0\} < 0\right\}$$

$$E\left[\sum_{t=1}^{B} \frac{\eta}{Y} \frac{\max\{R_t - C_t, 0\}}{(1+\delta_t)^t}\right] \geq \hat{A} + 1$$

$$Y, \eta \geq 0; \tag{17}$$

for simplicity, we assume that investors never sell their equity and only benefit from the dividends of profit sharing.

Figure 9 presents the results of numerical comparisons between revenue-sharing and equity crowdfunding contracts, where investors' opportunity cost $\hat{A} = 0.1$ and $r_t = \delta_t = 0.01$, $\forall t$. The left plot of Figure 9 presents the ratios z_E/z_F, where z_F (z_E) is the maximized NPV of revenue-sharing (equity) crowdfunding averaged over feasible Bolstr campaigns as a function of the volatility of cash flows. Because these ratios are less than one, they can be interpreted as the percentages of

the revenue-sharing NPV that are attained by the equity crowdfunding contract. We conclude that, if feasible, a firm can attain a higher NPV under a revenue-sharing contract than an equity crowdfunding contract. However, we note that four of 56 campaigns were feasible for high levels of uncertainty under equity crowdfunding but not under revenue-sharing crowdfunding. The right plot of Figure 9 presents the firm's bankruptcy probabilities for both revenue sharing and equity crowdfunding, again averaged over feasible Bolstr campaigns, as a function of the volatility of cash flows. We observe that the firm's bankruptcy probabilities under revenue-sharing contracts are within 0.1% of its bankruptcy probabilities under equity crowdfunding contracts. Furthermore, for low- and medium-cash flow volatility levels, revenue-sharing contracts have lower bankruptcy probabilities. In summary, similar bankruptcy probabilities and higher NPVs highlight the superiority of our proposed revenue-sharing contract over equity crowdfunding contracts.

6. Fixed Rate Loans

In this section, we analyze the performance of a standard fixed rate loan offered by other traditional and alternative lenders to serve as a benchmark for the revenue-sharing contract. We assume that these lenders offer fixed rate loans with flexible payment terms: the firm can decide on the investment amount Y (e.g., dollars) and the duration D (e.g., months) for a loan with a fixed interest rate of s per period (e.g., month). For simplicity in our analysis, we consider s to be constant with respect to D, and we show that, even for this conservative interest rate structure, revenue-sharing contracts result in higher NPVs for firms.

The loan payment per period is a standard amortization, and it is equal to $\frac{sY}{1-(1+s)^{-D}}$ (Stoft 2002). We

Figure 9. (Color online) Comparisons of Revenue-Sharing and Equity Contracts
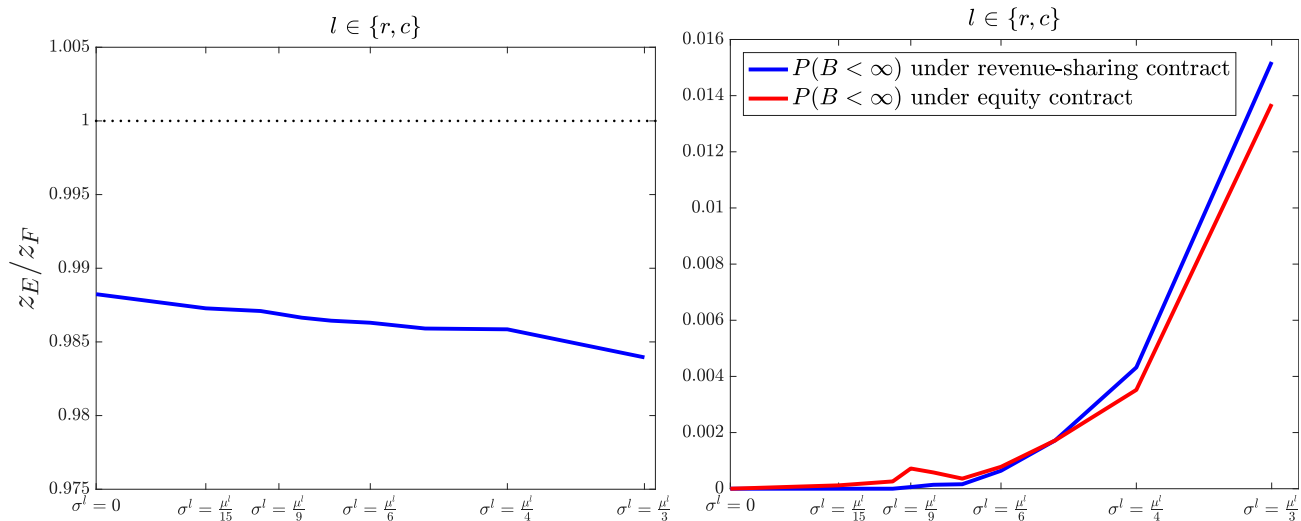
assume that the lender charges the borrower an origination fee of w percent of Y. The firm's maximization problem can be written as

$$z_L = \max_{Y,D} E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1 + r_t)^t}\right]$$
$$- E\left[\sum_{t=1}^{\min\{B,D\}} \frac{sY}{(1 - (1+s)^{-D})(1+r_t)^t}\right]$$
$$+ (1-w)Y$$
$$\text{s.t. } B = \min\left\{\hat{B} \geq 1 : \sum_{t=1}^{\hat{B}} (R_t - C_t) + (1-w)Y\right.$$
$$\left. - \sum_{t=1}^{\min\{\hat{B},D\}} \frac{sY}{1 - (1+s)^{-D}} < 0\right\}$$
(bankruptcy definition)
$$Y, D \geq 0. \quad (18)$$

Next, we solve the above problem numerically for Y and D. For this purpose, we need a base parameter set for the monthly interest rate s. For a conservative comparison, we consider $w = 0$. Interest rates and eligibility requirements vary across lenders. Banks usually have more strict eligibility requirements, but they offer relatively lower interest rates than some online lenders who have less stringent eligibility requirements. We consider monthly interest rates $s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\}$ for the numerical analysis.

We performed the numerical analysis for $D \in [1, 120]$ months and $Y \in [0, 2000000]$. We selected the upper bound of D to be 120 to account for long-term loans offered by the U.S. SBA and banks that give more flexibility to firms. The upper bound of Y is chosen so that it is consistent with the numerical analysis for Problem (6). Figure 10 presents the firm's maximized NPV under loans with monthly fixed rates $s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\}$ as a ratio to the firm's maximized NPV under revenue-sharing contracts, with

investors' opportunity costs $\hat{A} \in \{0.1, 0.2\}$. The ratios in Figure 10 are interpreted as percentages of the revenue-sharing NPV that are attained by the fixed rate loans. Therefore, a ratio less than one indicates that the firm's maximized NPV is higher under a revenue-sharing contract, and a ratio greater than one indicates that the firm's maximized NPV is higher under a fixed rate loan.

The firm's maximized NPV under a revenue-sharing contract with $\hat{A} \in \{0.1, 0.2\}$ is higher than loans with monthly fixed rates as small as $s = 0.07/12$. The firm's benefit from a flexible revenue-sharing contract is more significant when cash flow variability is high. For a revenue-sharing contract with $\hat{A} = \{0.1, 0.2\}$, the firm benefits from loans with a monthly fixed rate $s = 0.03/12$ but does not benefit from loans with higher rates, especially for higher levels of cash flow uncertainty. However, most loans offered to firms, especially small- to medium-sized firms, have relatively high interest rates. This underscores the importance of flexible payments that are possible under revenue-sharing contracts. Additionally, most low-rate loans, such as those offered by the SBA, have a processing time of 2–3 months. In contrast, according to marketing emails from Bolstr, firms can raise investments under revenue-sharing contracts within hours.

Figure 11 shows the average bankruptcy probabilities over feasible Bolstr campaigns under loans with monthly fixed rates $s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\}$ and under revenue-sharing contracts with investors' opportunity costs $\hat{A} \in \{0.1, 0.2\}$.

The bankruptcy probability for a firm under a revenue-sharing contract with $\hat{A} \leq 0.2$ is smaller than that of loans with monthly rates as small as $s = 0.14/12$. The highest bankruptcy probability is for a loan with monthly rate $s = 0.21/12$, and this again underscores the importance of flexible revenue-sharing contracts, especially for firms with higher levels of cash flow variability.

We conclude that firms benefit significantly more from revenue-sharing contracts than traditional fixed rate loans if investors' opportunity costs are not too

Figure 10. (Color online) Comparisons of Maximized NPV Under Revenue-Sharing Contracts and Fixed Rate Loans
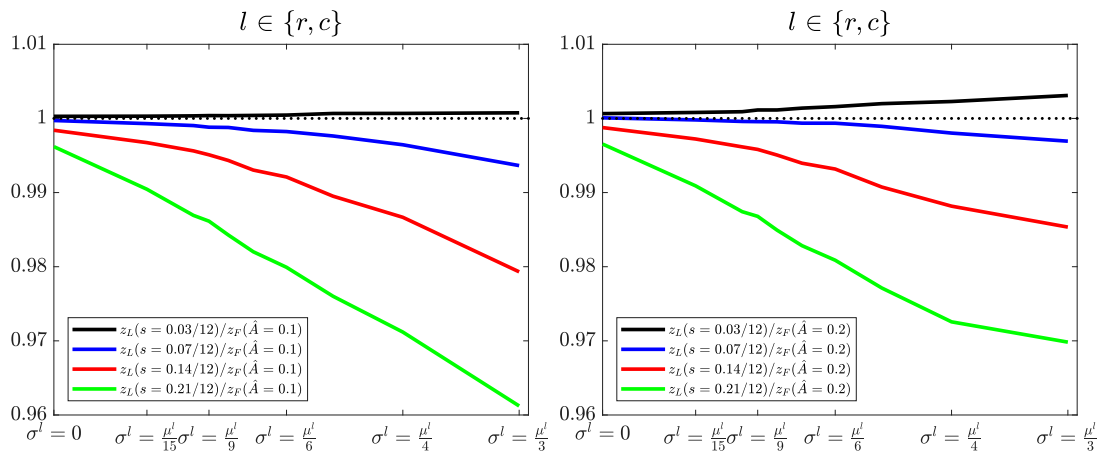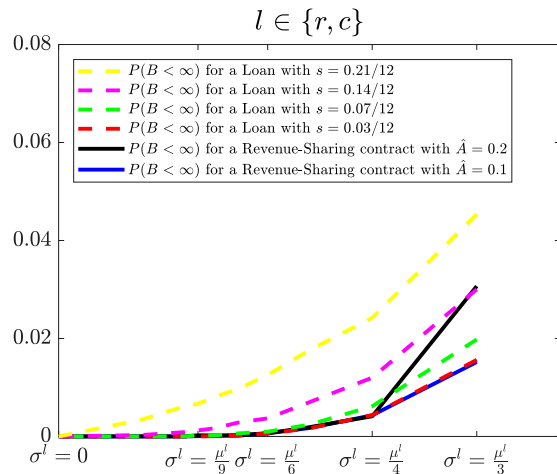
Figure 11. (Color online) Comparison of Average Bankruptcy Probabilities Under Revenue-Sharing Contracts and Fixed Rate Loans

large or if loans have nonnegligible interest rates. These results are of great importance to small firms, which usually have higher levels of cash flow uncertainty and need investment; according to a report by the SBA (U.S. Small Business Administration 2016), 73% of small firms used some type of financing in 2015–2016.

7. Conclusion

In this paper, we analyzed an emergent model of crowdfunding, in which a firm borrows capital and then pays back investors via revenue-sharing contracts. Specifically, the firm pays the investors a percentage of its revenues monthly until a predetermined investment multiple is paid over an uncertain investment horizon.

This paper is the first, to our knowledge, that studies this new model of crowdfunding. This model is facilitated by a platform (e.g., Bolstr, Localstake, or Startwise) that matches investors with a firm needing capital. This new model helps firms in need of investment to survive and thrive with a flexible contract that has terms that depend on the firm's performance. Indeed, when these contracts are used optimally, we provide evidence that the likelihood of firm bankruptcy is small, even for highly variable cash flows, because of the flexible monthly payments facilitated by the contract. If the revenue performance of the firm goes well, then the monthly payments to investors increase, which results in higher effective interest rates for investors. If revenue performance is poor, payments are lowered to reduce financial stress on the firm. We use real data from 56 Bolstr campaigns to motivate and calibrate our analytical models and to parameterize our numerical studies.

The first part of our paper formulates a stochastic programming model of the firm's expected NPV maximization problem with the contract parameters as variables for given values of the platform's origination

and servicing fees and investors' opportunity costs; unfortunately, this model is difficult to analyze. Therefore, in the second part of our paper, we formulate a deterministic approximation that we solve analytically. In the approximation problem, we use a cash buffer for dealing with cash flow uncertainties. In the third part, we evaluate the quality of the approximation solutions for the main stochastic model over feasible Bolstr campaigns for different levels of cash flow uncertainty. We see that the worst case average error over the campaigns is approximately 0.2%. Therefore, we conclude that the approximation problem provides high-quality solutions for the intractable stochastic problem.

In the final part of our paper, we compare revenue-sharing contracts with equity crowdfunding and fixed rate loans. We find that, for most cases considered, revenue-sharing contracts provide a higher NPV and a lower probability of bankruptcy than equity crowdfunding or a fixed rate loan. We also show that these benefits are more significant for firms with higher levels of cash flow uncertainty.

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