Permutations, Combinations and the Binomial Theorem

November 24, 2010

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Answer

This task can be performed in $27 \cdot 26 \cdot 25$ *different ways.*

Permutations and Sorting

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3. *The average number of inversions in a random permutation is the total number of inversions in all $n!$ permutations divided by $n!$.*
4. *But how can we find the total number?*

1. We shall count the total number of inversions in pairs.

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Remark

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So on the average, we'll have to perform $\frac{n(n-1)}{4}$ such exchanges.

Better sorting programs compare records that are far apart thus capable of removing more inversions in one exchange.

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Question

Can we design a sorting algorithm that will sort any given 5 objects in no more than 7 comparisons?

Question

For a fixed integer n what is the smallest number of comparisons a sorting algorithm needs to execute to sort any input list of n objects?

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We do have sorting algorithms that execute about $c \cdot n \log n$ comparisons.

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7. Cantor Digits: $n = \sum_{k=0}^{m} d_k \cdot k! \quad 0 \leq d_k \leq k.$

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Theorem

Every integer m has a unique representation:
$m = \sum_{k=0}^{s} d_k \cdot k! \quad 0 \leq d_k \leq k.$

Proof.

First recall that $\sum_{k=1}^{s} k \cdot k! = (s+1)! - 1$ so by the previous remark the representation is unique.

We now proceed by induction to prove that every integer has a Cantor Digits representation.

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6. $m + 1 = (d_k + 1) \cdot k! + \ldots$.

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Given an n- permutation $\pi = a_1 a_2 \ldots a_n$ we asociate with it the integer $f(\pi) = \sum_{k=1}^{n-1} d_k \cdot k!$.

The coefficients d_k are calculated as follows:

Let $a_j = k + 1$. Then $d_k = |\{a_{i_m} | i_m > j \textit{ and } (k+1) = a_j > a_{i_m}\}|$

In words: d_k is the number of entries in the permutation π that are to the right of $k + 1$ and are smaller than $k + 1$.

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Example

Let $\pi = 7\ 5\ 4\ 6\ 1\ 3\ 2\ 8$.

$d_1 = 0,\ d_2 = 1,\ d_3 = 3,\ d_4 = 4, d_5 = 3, d_6 = 6.$
So $f(\pi) = 6 \cdot 6! + 3 \cdot 5! + 4 \cdot 4! + 3 \cdot 3! + 2!$

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6. *In our example:* $f^{-1}(20000) = 7\ 1\ 6\ 5\ 8\ 3\ 4\ 2$.

Efficient Generation of Permutations and Combinations

Permutations can be generated either by the lexicographic order or by the Cantor-Digits enumeration.
There is another method called *The Arrow* algorithm.

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3. Stop when no arrow above an entry points to a smaller entry.

Example

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Remark (Generating Combinations)

We wish to generate all r-combinations of an n-set $\{a_1, a_2, \ldots, a_n\}$. *We shall proceed lexicographically:* $\{a_1, a_2, \ldots a_r\}$ *will be the first ("smallest") and* $\{a_{n-r+1}, \ldots, a_n\}$ *be the last ("largest").*

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Ans: $\{3, 6, 7, 8\}$.

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To simplify the notation, we shall assume that our universal set is $\{1, 2, \ldots, n\}$ and the numbers in the r subsets are sorted.

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Example

The 4-combination following the combination $\{3, 5, 7, 10\}$ in $\binom{\{1,2,\ldots,10\}}{4}$ is: $\{3, 5, 8, 9\}$.

The Binomial theorem

You probably know a few proofs of the classical binoial theorem:

Theorem

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There are many interesting relations among the binomial coefficieints. We shall briefly explore them and also see the technique of *double counting* used to prove many combinatorial identities. We start with Pascal's idenitity:

Theorem (Pascal's Identity)

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof.

Here is a simple combinatorial (double counting) proof:

□

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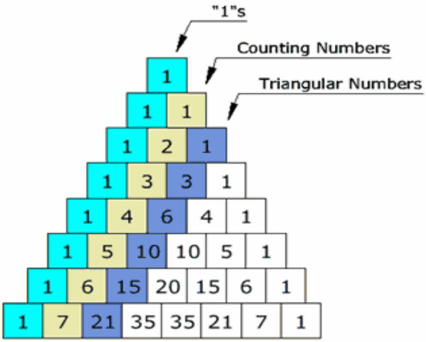
This relation among the binomial coefficient is traditionally encapsulated in the famous Pascal's triangle.

Pascal’s Triangle

Pascal’s Triangle contains many patterns and relations.



A Sample of Combinatorial Identies

There are literally thousands of combinatorial identities based on the binomial coefficients. We shall look at a small sample.

1. $$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

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2. $$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i-1}$$

   (or the number of ditinct subsets of even order is equal to the number of subset of odd order). Proof: $(1-1)^n = 0$.

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   Both sides count the number of ways to select a team of n students from a class with n male students and n females.

2. Vandermonde's Identity:

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A tribute to Gauss

Question

An urn contains 100 *balls numbered* $1, 2, \ldots, 100$. 100 *persons draw a ball, note the number on it and return it to the urn. What is the probability that no two persons draw the same ball?*

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Answer

There are 100^{100} *different ways to draw* 100 *balls. There are only* $100!$ *ways to draw different balls. So the probability that no two persons will draw the same ball is* $\frac{100!}{100^{100}}$. *So we need to estimate this number.*

Estimates

1. Simplest estimates:

$$n! = \prod_{i=1}^{n} i \leq \prod_{i=1}^{n} n = n^n \qquad n! = \prod_{i=1}^{n} i \geq \prod_{i=1}^{n} 2 = 2^n$$

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2. Slightly better estimates:

$$n! \geq \prod_{i=n/2}^{n} i \geq \prod_{i=n/2}^{n} n/2 = \left(\frac{n}{2}\right)^{\frac{n}{2}} \qquad n! \leq \left(\prod_{i=1}^{n/2} \frac{n}{2}\right)\left(\prod_{i=n/2}^{n} n\right) = \frac{n^n}{2^{\frac{n}{2}}}$$

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Remark

So the probability that each person will see a different number is $< 2^{-50}$ or just about no chance!

Even though it looks as if the estimates assume that n is even, it is not difficult to show that they hold for odd n.

Gauss' nice estimates

Theorem (Gauss)

$$n^{\frac{n}{2}} \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

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$i(n+1-i) \geq n \Rightarrow n! \geq \sqrt{n^n}$

□

We conclude by mentioning a very famous and beautiful approximation: Stirling's Formula.
It uses two of the most famous constants in mathematics: π and e in one expression involving an approximation of the integer valued function $n!$.

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

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Using Stirling's formula we get:

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20pt $\binom{2n}{n} \sim \frac{4^n}{\sqrt{2\pi}}$ Is another useful approximation.