Finite sets

October 7, 2010

Advanced problems on finite sets: Set Systems

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3. The first 2^n integers $\{0, 1, \dots 2^n - 1\}$ in binary are:
   $\{0_2, 1_2, 10_2, 11_2, \dots, 111\dots1_2\}$.
   Associate with every integer $n = b_1 b_2 \dots b_n$ the subset $\{k \ \text{if}\ \ b_k = 1\}$.

$\square$

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Also, if $B \in \mathbb{F}$ then $\overline{B} \notin \mathbb{F}$ therefore $\mathbb{F}$ can contain at most half the subsets of A.

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Let $A = \{a_1, a_2, \ldots, a_n\}$ and let
$\mathbb{F} = \{\{a_1, a_2\}, \{a_1, a_3\}, \ldots \{a_1, a_n\}, \{a_2, a_3, \ldots, a_n\}\}$.

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But can we have more than n subsets?

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Remark

As ususal, a closer look reveals that we can prove more. The same proof will work if we assume that all subset pairs have m members in common for some fixed m.

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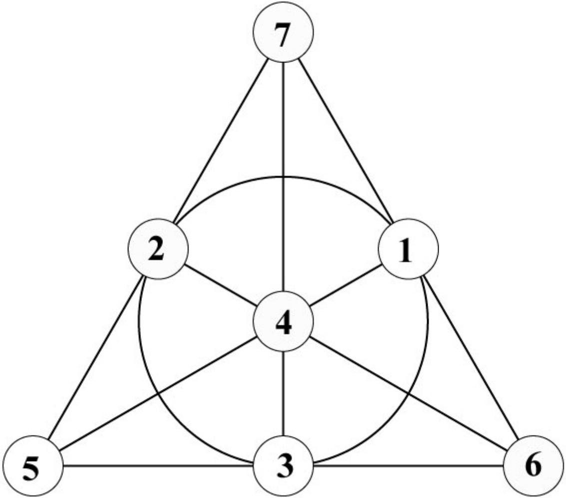

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3. It is easy to check that $\propto$ is an equivalence relation.

Construction of finite projective planes of order p^n

We start by defining the "points" of our projective geometry.

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The proof that these points and lines satisfy the definition of a finite projective plane will be included in the exercises following an example.

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It is now a simple matter to check that this set system satisfies the definition of a finite projective plane.

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7. Hence $\frac{|\mathbb{F}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \sum_{A \in \mathbb{F}} \frac{|A|! \times (n-|A|)!}{n!} \leq 1 \to |\mathbb{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

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We hope you enjoyed the journey.