How ”big” can a set be?

September 30, 2010

The cardinality of sets

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Observation

In this section we shall develop the tools that will enable us to compare sets. We will also prove that there are many non computable functions.

Functions

Definition

Let A and B be sets. A **function f** from A to B is an assignment of exactly one element of B to elements of A. Notation: $f : A \to B$.

Alternatively, $f \subset A \times B$ such that
$((a, b) \in f) \wedge ((a, c) \in f) \to b = c$.

In other words, a function $f : A \to B$ is a restricted binary relation between A and B.

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Definition

Let $f : A \rightarrow B, S \subset A,\ f(S) = \{b \in B | b = f(s)$ for some $s \in S\}$.

Examples

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1. f assigns to a bit string the number of 1s in the string.
Domain: $\{b|$ *All bit strings*$\}$ *Range* $= \{0, 1, 2, \ldots\} = N$.

Example

2. f assigns to each positive integer the smallest prime greater or equal to this integer.
$(f(5) = 5,\ f(25) = 29,\ f(69) = 71 \ldots$
Domain: Z^+, *Range the set of prime numbers.*

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Observation

The function $f(n) = 2n$ is a bijection between the integers and the even integers.
This means that there is a bijection between a set and "half" its size!

The inverse function

We need a few more definitions to be ready for our goal.

Definition

A set B is finite if there is a bijection between B and N_k.

Observation

If $f : A \to B$ is a bijection then we can define a new function $f^{-1} : B \to A$, the inverse of f, as follows: to find how f^{-1} maps the element $b \in B$ find the unique $a \in A$ such that: $f(a) = b$ and define $f^{-1}(b) = a$.

Example

$f(x) = 3x + 1, \; x \in R.$
$f^{-1}(x) = ?$

Definition

Let $g : A \rightarrow$B and $f : B \rightarrow C$. The **composition** of the functions f and g, denoted by $f \circ g$ is a function $f \circ g : A \rightarrow C$ defined by $f \circ g(a) = f(g(a))$.

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If f is a function on the set A, then $f \circ I(a) = I \circ f(a) = f(a)$.

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2. Let $h(x) = x^2 + 1$.

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$f \circ g(x)$ and $g \circ f(x)$ can be distinct functions, or the composition is not commutative.

The bijections on a set A form a group.

Theorem

If f, g, h are bijections on the set A then $(f \circ g) \circ h = f \circ (g \circ h)$

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In particular, if $|A| \geq |B| \wedge |B| \geq |A|$ *does it imply that* $|A| = |B|$*?*

Countable sets

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A set B is **countable** if $|B| = |N|$. We say that the cardinality of B is $\aleph_0$.

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- *If* $A \subset N,\ A \neq \emptyset$ *then* A *has a smallest member.*
- *(The axiom of mathematical induction).*
  If $1 \in A, \wedge ((n \in A) \rightarrow n + 1 \in A)$ *then* $A = Z^+$.

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- *(The axiom of mathematical induction).*
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Observation

There are other equivalent forms of the principle of mathematical induction:

1. $1 \in A, \ (\forall k < n, \ k \in A \rightarrow n \in A)$ then $A = Z^+$.

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If $A_i, \ i = 1, 2, \ldots$ *are countable sets then so is* $\cup_{i=1}^{\infty} A_i$.

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Corollary

There are functions $f : N \to \{0,1\}$ *(decision problems) that are not programmable.*

Theorem (4)

If $|A| \leq |B|$ *and* $|B| \leq |A|$ *then* $|A| = |B|$

Proofs

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Proof (Sketch of a proof for theorem 1)

We will prove that there is no onto function $f : A \to P(A)$.
Indeed given any function $f : A \to P(A)$. *Let*
$S = \{a \in A | a \notin f(a)\}$. *(Recall that* $f(a) \subset A$, *or* $f(a) \in P(A)$*).*

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Whether $s \in f(s)$ *or* $s \notin f(s)$ *we reach a contradicion.*
Fill in the details.
Conclusion: since there is an injection $g : A \to P(A)$ *and there is no otno function* $f : A \to P(A)$ *we conclude that* $|A| < |P(A)|$*.*

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Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ *be a countable set of real numbers. Let* $x_n = 0.x_{n,1}x_{n,2}\ldots x_{n,n}x_{n,n+1}\ldots$ *be the decimal expansion of* x_n*.*

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Let $y = 0.y_1y_2\ldots y_n\ldots$ *be defined as follows:*

Let $y_n = x_{n,n} + 5 \pmod{10}$. *We want to make sure that* $\forall n,\ y_n \neq x_{n,n}$.

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Remark

This proof technique is called the Diagonal Method. It is used on many occaisons. For instance Theorem 1 is an abstract form of this method.

Proofs

Here we go again.

Proof (Theorem 3, proof sketch)

It is enough to show that there is a bijection between the set of functions: $\{f : N \to \{0,1\}\}$ *and* $P(N)$.

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Show that this is a bijection between $P(n)$ *and the functions.*

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Proof (of the corollary)

Each program that implements a decision problem is stored in memory as a finite binary sequence. There are only countably many finite binary sequences. Hence there are non computable functions.

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Verify: Each chain is one of the following four types:

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Verify this assertion.

Surprise

Remark

There is a surprising consequence of this famous lemma. If you take two sets of points A and B in the plane, and if each set contains a disk, then each set can be disected into two sets A_1, A_2, B_1, B_2 *such that* A_i *and* B_i *are similar.*

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For example: these two sets can be disected into a pair of similar sets!