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Mr. Nguyen sells expensive jewelry. He has an interesting idea for a business model. Each customer will have access to boxes with a combination lock. Once a person grabs a box he can set his own private combination lock. An open box can be closed by anyone, but only the owner knows the combination and can open it. The content of any open box sent between persons will be stolen.
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You wish to buy an expensive gift for your significant other’s birthday. This means money will have to be sent to Mr. Nguyen (who is honest and trustworthy) and the gift delivered to you. Transaction details, such as item, price etc. can be discussed by phone. How can we accomplish this?
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Discussion

This is exactly how business transactions are being conducted on the Internet today, except that the boxes are virtual boxes. Closing a box is accomplished by encrypting the message. So while the message is traveling on the Internet, being exposed to hackers and others, it is encrypted using a “key”. Only the owner of the key knows how to open the box and retrieve its content.
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In 1976 Rivest, Shamir and Adelman proposed the public key cryptosystem: RSA.
The RSA public key system

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  - Private part: \( k = p \cdot q, \quad \gcd(e, (p - 1)(q - 1)) = 1 \).

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  \[ S = M^e \mod k \] and sends \( S \).
- Decryption: the receiver calculates \( S^d \mod k \) and retrieves \( M \) where \( d = e^{(-1)} \mod (p - 1)(q - 1) \).
$d = e^{(-1)} \mod (p - 1)(q - 1) \implies d \cdot e = a \cdot (p - 1)(q - 1) + 1$
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Security experts all over the world are trying hard to devise methods to factor large integers quickly. So far their efforts have not succeeded.
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We shall devote the rest of our time to take a quick glimpse at factoring.
Discussion

Note: we assume that everyone can intercept the message S. Furthermore, everyone knows exactly how S was calculated, everyone knows k and e, so why can’t they retrieve M?
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After all, all they need to do is calculate
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Our goal is to understand how this system works, why it is considered secure and other applications of this system.

To understand it we need to study some very mathematically interesting topics in modular arithmetic.
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An element $\alpha \in GF(q)$ is **primitive** if
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A polynomial $p(x)$ of degree $k$ over $GF(q)$ has at most $k$ roots.
To build an RSA cryptosystem we need to be able to test whether given numbers are prime and to “manufacture” large primes. We shall start by testing.

Question

Can Fermat's theorem be used for testing primality?

Answer

Unfortunately not. There are numbers for which the chances for finding an integer $a < n$ such that $a^{n-1} \mod n \neq 1$ are very slim. For instance if $n = (6k+1)(12k+1)(18k+1)$ and $(6k+1)$, $(12k+1)$ and $(18k+1)$ are prime, then if $\gcd(a, n) = 1$, $a^{n-1} \mod n = 1$. 

Discrete Mathematics Lecture-15
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Let $N$ be an integer. By Fermat's theorem if $N$ is prime then $a^{N-1} \mod N = 1$. This calculation can be executed very fast on integers with a few thousand digits. This means that if for some $1 < a < N - 1$; $a^{N-1} \mod N \neq 1$ then $N$ is definitely not a prime number.

But what can we conclude if $a^{N-1} \mod N = 1$?

Answer: NOTHING! $N$ may be prime and it may be composite! At best, we can try another integer $a$.

Example: As we noted in our drill, $k^{1728} \mod 1729 = 1$ for all $k$, $\gcd(k, 1729) = 1$. Our chances to randomly select $k$ such that $\gcd(k, 1729) > 1$ are very slim.
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In other words, the test fails to determine whether $N$ is composite. (Do you know another example of a “failing” test?)
Chứng minh.

If \( N \) is prime then \( w^{N−1} \mod N = 1 \) (Fermat).

So \( w^{(N−1)/2} \mod N = \pm 1 \).

If \( w^{(N−1)/2} \mod N = −1 \) the test stops. It is inconclusive.

If \( w^{(N−1)/2} \mod N = 1 \) we calculate \( w^{(N−1)/4} \mod N = \pm 1 \).

As long as the results of \( w^{(N−1)/2} \mod N = 1 \) we continue until we reach \( w^{2^k+1} \).

If \( w^{2^k+1} \mod N \neq \pm 1 \) then \( N \) is definitely composite.

We skip the important part of the proof: more than 50% of the integers \( a < N \) are composite-witnesses. So, to test whether an integer \( p \) is prime, randomly select 100 integers \( a < p \), apply to them the Miller-Rabin test. If the test fails, we assume that \( p \) is prime. The probability that we made a mistake, that is declared \( p \) is prime while it is not, is less than \((1/2)^{100}\) which is far less than the probability that the computer will make a mistake.
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- As long as the results of $w^{(N-1)/2^i} \mod N = 1$ we continue until we reach $w^{2^{k+1}}$.
- If $w^{2^{k+1}} \mod N \neq \pm 1$ then $N$ is definitely composite.
Chứng minh.

- Nếu $N$ là số nguyên tố thì $w^{N-1} \mod N = 1$ (Fermat).
- Vậy $w^{(N-1)/2} \mod N = \pm 1$.
- Nếu $w^{(N-1)/2} \mod N = -1$ thì test dừng lại, nó không xác định.
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- Ước số lẻ $w^{(N-1)/2^i} \mod N = 1$ chúng ta tiếp tục cho đến khi chúng ta đạt $w^{2k+1}$.
- Nếu $w^{2k+1} \mod N \neq \pm 1$ thì $N$ rõ ràng là số chẵn.
If \( N \) is prime then \( w^{N-1} \mod N = 1 \) (Fermat).

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We skip the important part of the proof: more than 50% of the integers \( a < N \) are composite-witnesses. So, to test whether an integer \( p \) is prime, randomly select 100 integers \( a < p \), apply to them the Miller-Rabin test. If the test fails, we assume that \( p \) is prime. The probability that we made a mistake, that is declared \( p \) is prime while it is not, is less than \( \left(\frac{1}{2}\right)^{100} \) which is far less that the probability that the computer will make a mistake.
Example

1729 is a composite integer. 

\[ 1728 = 2^6 \cdot 3^3. \]

\[ 3^2 \cdot 3^3 = \mod 1729 = 1 \text{ for } 1 \leq i \leq 6. \]

But \[ 3^3 = 664 \] proving that 1729 is composite.

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- RSA-1024 (2¹⁰ bits or 309 decimal digits) has not been factored. There is a $100,000 USD prize offered for its factorization. It is of particular interest as this is the current size used in applications.
- RSA-2048 (617 decimal digits) has a prize of $200,000 USD for its factors.
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$r \in \mathbb{GF}^*(q)$ is a quadratic-residue mod $q$ if there is an $s \in \mathbb{GF}(q)$ such that $s^2 = r$.

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- **Calculate** \( a = n^{2k+1} \mod p \).
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- Calculate \( a = n^{2k+1} \) mod \( p \).
- If \( a = -1 \) stop, \( n \) does not have a square root mod \( p \).
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  *then* $\sqrt{a} \mod p = a^{s+1} \cdot b^t \mod p$.
- *This can be accomplished as follows:*
While $a^{2^d(2m+1)} \mod p = 1$ do: $d = d - 1$. 

For an example see the SAGE sample in the supplements folder.
While \( a^{2d(2m+1)} \mod p = 1 \) do: \( d = d - 1 \).

This loop will terminate either when \( a^{2d(2m+1)} \mod p = -1 \) or \( a^{2m+1} \mod p = 1 \).
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This is easy. Note that $b^{2^{k-1}(2m+1)} \mod p = -1$ so:
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If \( a^{2^{m}+1} \mod p = 1 \) then \( \sqrt{a} \mod p = a^{m+1} \mod p \).

If \( a^{2^{d}}(2^{m}+1) \mod p = -1 \) then find \( b \), a non-quadratic residue \( \mod p \).

This is easy. Note that \( b^{2^{k-1}}(2^{m}+1) \mod p = -1 \) so:

\[ a^{2^{d}}(2^{m}+1)b^{2^{k-1}}(2^{m}+1) \mod p = 1 \]

We can repeat reducing the exponent by a factor of 2, multiplying by \( b \) to make sure that the product will remain 1 until we reach \( a^{2^{k+1}}b^{2j} \mod p = 1 \).

\( \sqrt{n} \mod p = a^{k+1}b^{j} \mod p \).

For an example see the SAGE sample in the supplements folder.
See the file factoring.pdf