“Named” numbres

Ngày 25 tháng 11 năm 2011

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The reason they are “named” is because they appear in many forms in mathematics and other sciences.

Stirling Numbers

Stirling Numbers are named after the Scottish mathematician James Stirling who introduced them in the 18^{th} century. There are two kinds of Stirling numbers (with various notation):

$$\text{Stirling numbers of the first kind: } \left[\begin{matrix} n \\ k \end{matrix} \right] = c(n, k)$$

$$\text{Stirling numbers of the second kind: } \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = S(n, k)$$

Both numbers describe combinatorial counting that lead to a “triangular” recurrence relation similar to the binomial coefficients.

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$\{\{a\}, \{, b, c, d, e\}\}, \{\{b\}, \{a, c, d, e\}\} \ldots, \{\{a, b\}, \{c, d, e\}\}, \ldots$
$\{\{d, e\}, \{, a, b, c\}\}$. *For a total of:*

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$\{\{d, e\}, \{, a, b, c\}\}$. *For a total of:* $5 + 10 = 15$ *different partitions.*

Definition

$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ *The Stirling number of the second kind is the number of ways to partition an n-set into k non-empty subsets.*

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As for the set B for every partition $B \setminus \{x_0\}$ into k subset we can add x_0 to any one of the k parttions yielding:

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\}$$

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This "*triangular*" relation is very similar to Pascal's identity for binomials.

An application

The polynomials of degree n form a vector space over the field $\mathbb{R}$, and so do the polynomials $\{1, x, x(x-1), x(x-1)(x-2)\ldots\}$. A common notation for $x(x-1)\ldots(x-j+1)$ *is* $x^{\underline{j}}$

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What are the coefficients a_i ?

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$$x^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}}$$

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$$x \cdot x^{n-1} = x \sum_{k=0}^{n-1} \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} x^{\underline{k}} = \sum_{k=0}^{n-1} \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} (x^{\underline{k+1}} + kx^{\underline{k}}) =$$

$$\sum_{k=0}^{n-1} \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} kx^{\underline{k}} + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} x^{\underline{k}} = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^{\underline{k}}$$

Follows from the triangular relation.

□

The Stirling numbers of the first kind are defined by a closely related relation:
It counts in how many ways you can arrange n objects into k disjoint cycles. So for example, the partitions $\{[1,3][2,5,4]\}$ and $\{[1,3][2,4,5]\}$ are distinct but $\{[3,1],[5,4,2]\}$ is the same as the first partition.

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4. $$\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right] = n!$$