Discrete Optimization
Lecture-9

Ngày 22 tháng 10 năm 2011

Hall's Theorem

First we have an equivalent formulation of Hall's theorem:

Theorem

Let G be a bipartite graph with partitions X, Y. Let $N(x) = \{y \mid (x, y) \in E(G)\}$. There is a mtching M that saturates X if and only if $\forall A \subset X, \mid \bigcup_{x \in A} N(x) \mid \geq |A|$.

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An immediate corollary is yet anohter proof that a $k-$ regular bipartite graph has a perfect matching.

Let us re-visit the previous example.

Let us introduce a slight change:

$\{1,3,2,5\},\{1,3,4\},\{1,4,8\},\{2,3,5,6\},\{2,4,6\},\{1,5,2,\}$
$\{1,3,7\},\{1,4,5,6\}$ *?*

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But we can find an augmenting path!
$\{1,4,5,6\} \leftarrow 1 \leftarrow \{1,3,2,5\} \leftarrow 3 \leftarrow \{1,3,4\} \leftarrow 4 \leftarrow \{1,4,8\} \leftarrow 8$
is an augmenting path! So here is the SDR:
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Watch the blackboard! □

More matching theory and applications

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Let $A = (a_{i,j})$ *be a matrix with* $a_{i,j} \in \mathbb{N}$ *such that:*

$$\sum_{i=1}^{n} a_{i,j} = m \ \ \forall 1 \leq j \leq n \text{ and } \sum_{j=1}^{n} a_{i,j} = m \, \forall 1 \leq i \leq n$$

Then $A = P_1 + P_2 + \ldots = P_m$ *where* P_i *are permutation matrices.*

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Definition

1. *An* $n\times n$ *real, non-negative matrix is called* **doubly stocahstic** *if the sum of every row and every column is* 1*.*
2. *The expression* $\sum_{i=1}^{n}\alpha_i v_i$ *where* v_i *are vectors* $\alpha_i\geq 0,\quad \sum_{i=1}^{n}\alpha_i=1$ *is called a* **convex combination** *of the* $v_i's$*.*

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The proof of this important theorem is similar to the proof of the integer case and left to you.