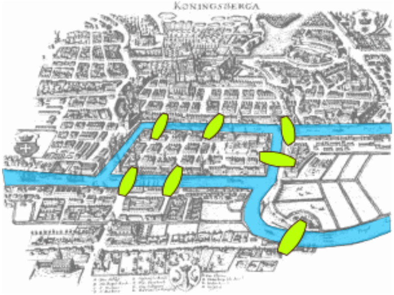
Discrete Optimization
Lecture-8

Ngày 17 tháng 10 năm 2011

Eulerian Cycles

The birth of graph theory is attributed to Leonard Euler. Euler was asked to solve a puzzle that preoccupied the citizens of Königsberg. The people wondered if they could start at some region, cross all bridges exactly once and end up where they started.



The seven bridges on the Pregel river

The Königsberg graph

Answer

Euler constructed a multi-graph whose vertices are the four regions determined by the river and added edges between two regions for every bridge connecting them.

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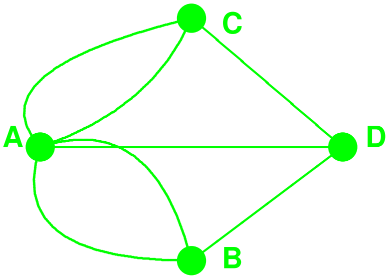

The Königsberg graph

Comment

If a graph has an Eulerian cycle, every time we visit a vertex we exit on a different edge. This means that the degree of every vertex must be even. Euler's graph has four vertices, seven edges. The degrees of the vertices are $(5, 3, 3, 3)$*. So clearly it is not possible to walk through all edges exactly once even if you do not insist to return to your starting region.*

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Theorem

A multi-graph $G(V, E)$ *is* **Eulerian** *if and only if it is connected and every vertex has even degree.*

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A digraph $D(V, E)$ *is* **Eulerian** *if and only if it is strongly connected and* $\forall v \in V\ d_{in}(v) = d_{out}(v)$.

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Note that it can be efficiently checked and actually construct an Eulerian cycle if it exists. As we shall see later, detecting a hamiltonian cycle in a graph is one of the most algorithmically difficult problems.

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Matchings

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On board □

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On board □

Remark

A perfect matching in a graph $G(V, E)$ is also a vertex cover so $\nu(G) = \rho(G) = \frac{|V(G)|}{2}$.

For an odd cycle $C_{2k+1},\ \nu(C_{2k+1}) = k,\ \rho(C_{2k+1}) = k + 1$.

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Remark

Clearly, if P *is an* $M-$*augmenting path then* $M \Delta E(P)$ *is a matching with* $\mid M \mid +1$ *edges. That is* M *is not a matching of largest size.*

Augmenting paths are essential tools in studying matchings in graphs.

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As noted previously, if G has an $M-$ augmenting path then there is a bigger matching.

To prove the opposite, assume that there is a bigger matching N. We look at the subgraph spanned by $M \cup N$. It is 2-edge colorable and the degrees of its vertices are 1 or 2. So its connected components are even cycles containing the same number of edges from M and N and paths. Since $\mid N \mid > \mid M \mid$ there must be a path starting and ending in edges from N and this is an M-augmenting path. □

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- *Trees are bipartite graphs.*
- *G is bipartite iff every connected component of G is bipartite.*

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A 2-coloring of a bipartite graph G partitions the vertices into two independet sets A and B called partitions.

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In class on the board □

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This was a key fact in the Hungarian method. Unfortunately the proof of Kőnig's theorem does not shed a light on how to find the minimal set of lines or why our algorithm works, another proof is needed for that. But at least it justifies our claim.

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As noted before, $\mid V(G) \mid= 2n$ and $\mid E(G) \mid= nk$. Any set A can cover at most $\mid A \mid k$ edges so a vertex cover must have at least n vertices. A partition covers all edges hence $\tau(G) = \nu(G) = n$ and the maximum matching is a perfect matching. □

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Corollary

A k-regular bipartite graph G is k-edge colorable ($\gamma_1(G) = k$).

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Let $\{A_1, A_2, \ldots, A_k\}$ *be subsets of the finite set A. A set* $\{a_1, a_2, \ldots, a_k\}$ *is called a* **a system of distinct representatives (SDR) (transversal)** *if* $a_i \in A_i$.

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Example

Does the following family of subsets have an SDR?
$\{1, 3, 2, 5\}, \{1, 3, 4\}, \{1, 4, 7\}, \{2, 3, 5, 6\}, \{2, 4, 6\}, \{1, 5, 2, \}$
$\{1, 3, 7\}, \{1, 4, 5, 6\}$ *?*

Hall's Theorem

Theorem

A necessary and sufficient condition for $\{A_1, A_2, \ldots, A_k\}$ *to have an SDR is:* $\forall J \subset \{1, 2, \ldots, k\} \;\; |\bigcup_{j \in J} A_j| \geq |J|$.

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The condition is obviously necessary. If $A_{i_1}, A_{i_2}, \ldots, A_{i_j}$ have a system of distinct representatives then $\bigcup_{r=1}^{j} A_{i_r}$ must contain at least j members.

To prove the sufficiency, we build a bipartite graph with one partition the set $A = \bigcup_{i=1}^{k} A_i$ and the other partition B with k vertices corresponding to the different k subsets A_i.

We connect b_r by an edge to a_j if $a_j \in A_r$.

We proceed to show that if M is not a maximum matching then we can find an augmenting path. □