Discrete Optimization
Lecture-10

Ngày 7 tháng 10 năm 2011

Linear Programming

Question

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The generic form of a LP problem is:

$$\text{Maximize:} \quad \sum_{i=1}^{n} c_i x_i \tag{1}$$

$$\text{Subject to:} \quad \sum_{i=1}^{n} a_{i,j} x_i \leq b_j \quad j = 1, 2, \ldots m \tag{2}$$

$$x_j \geq 0, \;\; c_i, \;\; a_{i,j}, \;\; b_j \in \mathbb{R} \tag{3}$$

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Equation (1) is the **The Objective Function**

Equations $(2-3)$ are the **constraints.**

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In practice, Linear Programs with hundreds of thousands constraints and variables are used by many applications.

Finding efficeint ways to solve such huge problems is an active research topic.

(A brief history)

Linear Programming was first developed by the Russian mathematician Leonid Kantarovich in 1939 to help the war effort which had many logistic problems. The simplex method was developed by George Danzig in 1949. *It is considered one of the top ten most important algorithm developed in the* 20^{th} *century. Danzig was an American mathematician working in Stanford University, California.*

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There are many implementations of the simplex method in the market. We installed in our lab a program called AIMMS which we shall use in this class.

Exampe: The assignment problem

Example

Let $c_{i,j}$ be the amount bid by company j to perform job number i. We introduce the so-called decision variables *$x_{i,j}$ defined as:*

$$x_{i,j} = \begin{cases} 1 & \textit{If job number i is assigned to company number j.} \\ 0 & \textit{otherwise.} \end{cases}$$

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The corresponding linear programming problem is:

$$\begin{aligned} \textit{Minimize:} \quad & \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j} x_{i,j} && (4) \\ \textit{Subject to:} \quad & \textstyle\sum_{i=1}^{n} x_{i,j} = 1 \qquad j = 1, 2, \dots n && (5) \\ \textit{and} \quad & \textstyle\sum_{j=1}^{n} x_{i,j} = 1 \qquad i = 1, 2, \dots n && (6) \\ x_{i,j} \geq 0 & && (7) \end{aligned}$$

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This brings us to the Integer Linear Programming problems:

Definition (**Integer Linear Programming (ILP)**)

$$\textit{Maximize:} \quad \sum_{i=1}^{n} c_i x_i \tag{8}$$

$$\textit{Subject to:} \quad \sum_{i=1}^{n} a_{i,j} x_i \leq b_j \quad j = 1, 2, \ldots m \tag{9}$$

$$x_j \geq 0, \quad x_i \textit{ integers.} \tag{10}$$

LP and ILP

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At first sight, it seems like ILP problems are easier than LP problems. After all, for an ILP problem there will be only a finite number of feasible solutions, so we can enumerate them all and find the optimal. On the other hand, in the general LP problem there are infinitely many feasible solutions so we cannot enumerate them.

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This raises another question, when can we expect the solution to an LP problem to be a vector of integers? For instance, we will be able to prove that the LP solution to the assignment problem is guranteed to be an integer!

Example: Bia Hà Nội

Discussion

15 *breweries need to supply beer to* 35 *retaurants around Hà Nội. Each brewery can supplly* s_i *barrels of beer per day. Each restaurant needs* d_j *barrels per day. The cost of delivering a single barrel from brewery* i *to restaurnt* j *is* $c_{i,j}$.

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$$\text{and} \quad \sum_{j=1}^{35} x_{i,j} = s_i \qquad i = 1, 2, \ldots n \tag{13}$$

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Note that a solution exists only if $\sum_{i=1}^{15} s_i = \sum_{j=1}^{35} d_j$.

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- *While it is not clear that the optimal solution to the LP problem will yield values of* 0 *or* 1 *to the decision variables, we will be able to prove that this is the case.*
- *It is noteworthy that it is actually a matching problem. Given an* $n \times n$ *complete weighetd bipartite graph, we wish to find a perfect matching of smallest cost.*

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$$\begin{aligned} \text{Minimize:} \quad & \sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j} c_{i,j} & (15) \\ \text{Subject to:} \quad & \sum_{i=1}^{n} x_{i,j} = d_j & (16) \\ & \sum_{j=1}^{m} x_{i,j} = s_i & (17) \\ & x_{i,j} \geq 0 & (18) \end{aligned}$$

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- *While it is known that in some cases the simplex method can run in exponential time and there are other algorithms that run in polynomial time, in reality, the simplex method exhibits the best running time.*
- *We shall study in this class the implementation that is easiest to follow and understand and leave the speedier implementations for those of you that will need to use it in the future.*

Definition (**The standard form)**

The following linear program is in the standard form :

$$\begin{aligned} \textit{Maximize:} \quad & \textstyle\sum_{i=1}^{n} c_i x_i && (19)\\ \textit{Subject to:} \quad & \textstyle\sum_{i=1}^{n} a_{i,j} x_i = b_j \quad j = 1, 2, \ldots m && (20)\\ x_j \geq 0. & && (21) \end{aligned}$$

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- *A constraint of the form* $\sum_{j=1}^{k} a_{i,j} x_j \leq b_i$ *can be changed to:* $\sum_{j=1}^{k} a_{i,j} x_j + s_i = b_i \quad s_i \geq 0$

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- *And if a variable* x_i *is unresitricted we can replace every occurence of* x_i *by the difference* $x_{i_1} - x_{i_2}$ *so that all variables remain restricted to non-negative values.*

The standard form

Example (Let the given LP problem be:)

$$\textit{Minimize:} \quad 3x_1 - 4x_2 + x_3 \tag{22}$$

$$\textit{Subject to:} \quad x_1 + 2x_2 - x_3 \leq 5 \tag{23}$$

$$2x_1 + 3x_2 + x_3 \geq 4 \tag{24}$$

$$2x_1 - 3x_2 - x_3 = -1 \tag{25}$$

$$x_1 \textit{ unrestricted, } x_2, x_3 \geq 0 \tag{26}$$

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& 2(x_{1_1} - x_{1_2}) - 3x_2 - x_3 = -1 && (30)
\end{aligned}$$

The standard form

Example (Let the given LP problem be:)

$$\text{Minimize:} \quad 3x_1 - 4x_2 + x_3 \tag{22}$$

$$\text{Subject to:} \quad x_1 + 2x_2 - x_3 \leq 5 \tag{23}$$

$$2x_1 + 3x_2 + x_3 \geq 4 \tag{24}$$

$$2x_1 - 3x_2 - x_3 = -1 \tag{25}$$

$$x_1 \text{ unrestricted, } x_2, x_3 \geq 0 \tag{26}$$

$$\text{Maximize:} \quad -3(x_{1_1} - x_{1_2}) + 4x_2 - x_3 \tag{27}$$

$$\text{Subject to:} \quad (x_{1_1} - x_{1_2}) + 2x_2 - x_3 + s_1 = 5 \tag{28}$$

$$2(x_{1_1} - x_{1_2}) + 3x_2 + x_3 - s_2 = 4 \tag{29}$$

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$$x_{1_1}, x_{1_2}, x_2, x_3, s_1, s_2 \geq 0 \tag{31}$$

How does the simplex method work?

We shall follow the steps of the simplex method on the following example (taken from V. Chvátal's book):

$$\text{Maximize:} \quad 5x_1 + 4x_2 + 3x_3 \tag{32}$$

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$$4x_1 + x_2 + 2x_3 \leq 11 \tag{34}$$

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Rewrite the standard form as:

$$z - 5x_1 - 4x_2 - 3x_3 = 0 \tag{37}$$

$$s_1 = 5 - 2x_1 - 3x_2 - x_3 \tag{38}$$

$$s_2 = 11 - 4x_1 - x_2 - 2x_3 \tag{39}$$

$$s_3 = 8 - 3x_1 - 4x_2 - 2x_3 \tag{40}$$

$$\text{Maximize z subject to: } x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \tag{41}$$

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- The optimal value of z determined by equations $(37 - 41)$ will give us an optimal solution for the original LP problem.
- If we can increase the values of x_1, x_2, x_3 in equations $(37 - 41)$ the value of z will also increase.

The simplex method

- We can start with the initial feasible solution $x_1 = x_2 = x_3 = 0,\ s_1 = 5,\ s_2 = 11,\ s_3 = 8$ which yields $z = 0$.

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- Yes there is a limit. For instance, if $x_1 = 3$ equation (36) cannot be satisfied with $s_3 \geq 0$.

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- We notice that the more we increase the value of x_1 the higher the value of z will be. But is there a limit to the size of the inrease?
- Yes there is a limit. For instance, if $x_1 = 3$ equation (36) cannot be satisfied with $s_3 \geq 0$.
- It is now clear that x_1 cannot be greater than $\frac{5}{2}$ getting the feasible solution: $x_1 = \frac{5}{2}, x_2 = x_3 = 0,\ s_1 = 0,\ s_2 = 1,\ s_3 = \frac{1}{2},\ z = \frac{25}{2}$, a big improvement.

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$$z - 5(\frac{5}{2} - \frac{1}{2}s_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3) - 4x_2 - 3x_3 = 0 \quad (42)$$

$$x_1 = \frac{5}{2} - \frac{1}{2}s_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \quad (43)$$

$$s_2 = 11 - 4(\frac{5}{2} - \frac{1}{2}s_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3) - x_2 - 2x_3 \quad (44)$$

$$s_3 = 8 - 3(\frac{5}{2} - \frac{1}{2}s_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3) - 4x_2 - 2x_3 \quad (45)$$

The simplex method

After rearranging the terms on the right side of equations $(42-45)$ we get:

$$z = \frac{25}{2} - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}s_1 \tag{46}$$

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}s_1 \tag{47}$$

$$s_2 = 1 + 5x_2 + 2s_1 \tag{48}$$

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Repeating the same process we get:

The simplex method

$$z = 13 - 3x_2 - s_1 - s_3 \tag{50}$$

$$x_3 = 1 + x_2 + 3s_1 - 2S_3 \tag{51}$$

$$x_1 = 2 - 2x_2 - 2s_1 + s_3 \tag{52}$$

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The feasible solution is:

$$x_1 = 2, x_2 = 0, x_3 = 1, s_1 = 0, s_2 = 1, s_3 = 0. \tag{54}$$

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Can you see why?

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This has been a very brief review of the simplex method. The study of Linear Programming is usually divided into the following parts:

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5. ILP (Integer or mixed integer Linear Programs).

In this class we shall study how to use LP to solve practical problems. AIMMS will take care of the technical details for us.