Kruskal's MCST algorithm

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Theorem 1. *Kruskal's algorithm results in a minimum cost spanning tree for the wieghted graph $G(V,E)$.*

Proof.

1. Let $e_1, e_2, \ldots, e_{|E|}$ be the edges of G arranged in increasing order of their weights.
2. Let $e_{i_1}, e_{i_2}, \ldots, e_{e_{|V|-1}}$ be the edges of the spanning tree K selected by Kruskal's algorithm.
3. Let T be a minimum cost spanning tree.
4. Among all MCST's let e_{i_m} be the largest index m such that $\{e_{i_1}, e_{i_2}, \ldots, e_{i_m}\} \subset E(T)$.
5. We will show that $i_m = |V| - 1$.
6. If not, add e_{m+1} to T. This will create a cycle in it.
7. At least one edge of this cycle is not included in $\{e_{i_1}, e_{i_2}, \ldots, e_{i_m}\}$.
8. Let this edge be e_j.
9. Since $e_j \notin \{e_{i_1}, e_{i_2}, \ldots, e_{i_{m+1}}\}$ it means that either $j > i_{m+1}$ or Kruskal rejected it because it created a cycle with the edges $\{e_{i_1}, e_{i_2}, \ldots, e_{i_m}\}$.
10. But $\{e_{i_1}, e_{i_2}, \ldots, e_{i_m}\} \subset E(T)$ so we must have $j > i_{m+1}$
11. But this means that $\omega(e_j) \geq \omega(e_{m+1})$ so if we add e_{m+1} to T and remove e_j from it we cannot decrease the cost since T was a MCST.
12. This can only mean that the new tree has $m+1$ edges in common with K.
13. This implies that K is MCST.

□