Discrete Optimization
Graphs

Ngày 20 tháng 9 năm 2011

Definition (Isomorphism)

Two graphs $G(V, E)$ and $H(V, E)$ are **isomorphic** *if there is a bijection $\phi : V(G) \rightarrow V(H)$ such that $(x, y) \in E(G)$ if and only if $(\phi(x), \phi(y)) \in E(H)$*

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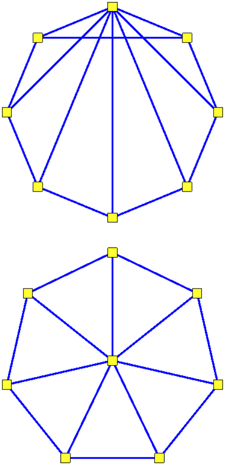
Question

Is it easy to decide whether two graphs are isomorphic?

Isomorphism Example



Are these graphs isomorphic?

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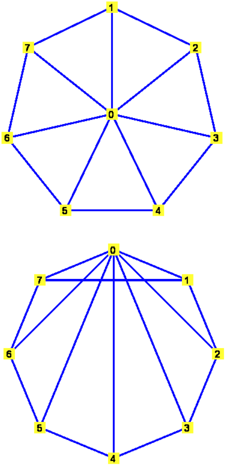

Two isomorphic copies of W7

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As with the assignment problem, this approach is not practical.

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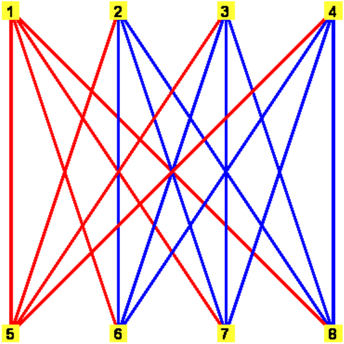
Lemma

Every connected graph has a **spanning tree.**

Spanning and Induced Subgraphs



A graph of order 8 (eight vertices)
The red edges form a spanning subgraph.
The blue edges form an induced subgraph.

The Complement and Line Graphs

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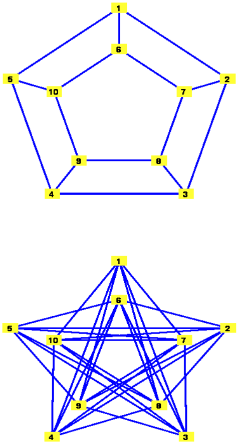
Definition (**Line Graph**)

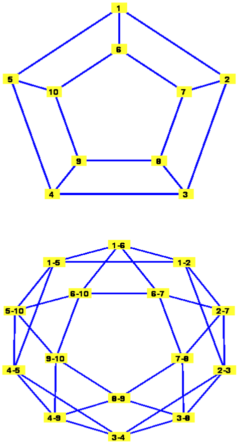
The **line graph** *$L(G)$ of a graph G is a graph with $V(L(G)) = E(G)$ and two vertices in $L(G)$ are connected by an edge if the two corresponding edges in G share a vertex.*

The 5-prism and its complement



The 5-prism and its complement.

The 5-prism and its line-graph



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(Graph Coloring)

Assigning labels to vertices or edges of graphs is a fundamental tool in many proofs and algorithms on graphs. We call this procedure **coloring**. *In various applications we use different rules for coloring.*

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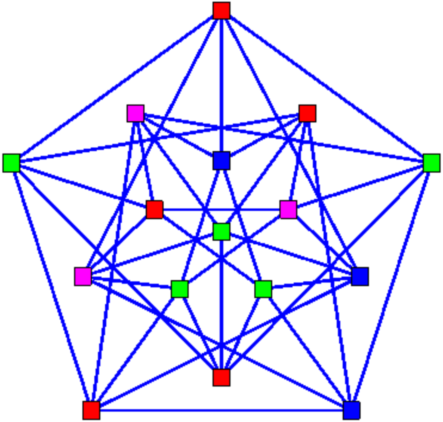
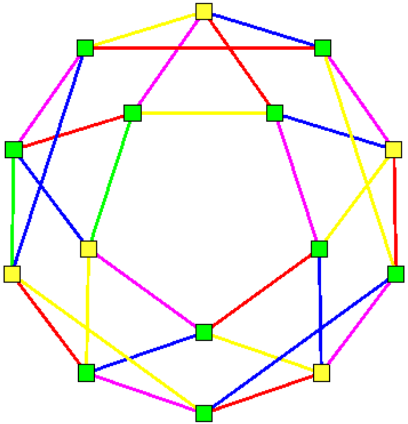
Vertex Coloring



Clebsch graph: A 5-regular graph with chromatic number 4.

Edge Coloring



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Comment

Many of these graph parameters are inter-related. In many applications we need to calculate them. Some can be calculated effciently but

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There are many useful implications to these relations. For instance, if you find a matching and a vertex cover of the same size, then the matching is a maximal matching and the cover is a minimal cover.

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Or if there is an "efficient" algorithm to find the size of the largest clique in a graph then there is an efficient way to find $\alpha(G)$.