Discrete Mathematics
Drill

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Hanoi 2011
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Tuesday Nov. 29

1 Vizing's Demo execution

1. For convenience and future use in the general proof we number the colors as:
   red - 1, magenta - 2, blue - 3, green-4, cyan - 5.

2. The general strategy is to find a path $v_1 - v_2 - \ldots - v_k$ which is colored it two colors i, j such that no edge at vertex v_1 is colored j and no edge at the end-vertex v_k is colored i.

3. If we can find such a path, then we can switch the colors along the path from i to j and from j to i obtaining a new proper edge coloring.

4. Note that the 3 edges incident with 4 are colored $\{1, 2, 3\}$ and three edges incident with 5 are colored $\{2, 4, 5\}$ so there is no color available for the missing edge $(4, 5)$.

5. Our strategy is to change the colors on some of the edges incident with 4 and construct a path as mentioned above so that at the end of the changes there will be a color available for the edge $(4, 5)$.

6. We begin with listing potential re-coloring of edges incident with 4:

7.
   - To each vertex i we assign a "color" $x(i)$ such that this color is not included among the colors of the edges incident with i.
   - The vertices incident with 4 are $\{1, 8, 9\}$. Looking at the given coloring we see that $x(4) = 5,\ x(1) = 5,\ x(8) = 3,\ x(9) = 2$.
   - $x(5) = 1$. Note: we actually have two choices for $x(5)$ our "algorithm" could have chosen $x(5) = 3$.

8. Let us start with the first choice: $x(5) = 1$. If there is no edge colored 1 incident with 4 we can color the edge $(4, 5)$ with 1.
9. But there is such an edge: $(4, 1)$.
10. We are lucky! $x(1) = 5$. This means that we can change the color of the edge $(4, 1)$ to 5 and color the edge $(4, 5)$ red (1).
11. Let us see what can we do if we were not so lucky and our algorithm chose $x(5) = 3$.
12. We check whether there is an edge $(4, i)$ with $x(i) = 3$, indeed $x(8) = 3$. We proceed by looking for an edge $(4, j)$ for which $x(j) =$ color of $(4, 8)$.
13. The color of $(4, 8) = 2 = x(9)$. The color of $(4, 9)$ is 3 and $x(1) = 5 \neq 3$ so we stop.
14. We first change the colors of $(4, 8)$ to blue (3) and of $(4, 9)$ to magenta (2). We shall now start to build the path along which we will change the colors. The path will start with 4 at which the color 5 is missing and alternate between magenta (2) and cyan (5).
15. The path is: $4 - 9 - 15 - 6 - 14 - 5 - 3 - 1$. It ends at 1 where $x(1) = 5$.
16. We change the colors along this alternating path and obtain a coloring of $G - (4, 5)$ in which the color blue (3) is missing at both 4 *and* 5.
17. so $\chi_1(G) \leq 5$.

The general proof: (by induction).

1. Let $G_1 = G - (a, b)$ be colored by $\Delta(G) + 1$ colors:

   $\phi : E(G) \longrightarrow \{1, 2, \ldots \Delta(G) + 1\}$.
2. For each $v \in V(G_1)$ let $x(v) = i$ such that $i \notin \{\phi(v, u) \ \mid \forall u \in N_{G_1}(v)\}$.
3. Note that such an i exists since $\mid N_{G_1}(v) \mid \leq \Delta(G)$.
4. Let $N_G(a) = \{v_0 = b, v_1, \ldots v_{k-1}\}$.
5. Let $v_{i_1} \in N_G(a)$ be such that $\phi((a, v_{i_1})) = x(b)$. Note that if there is no such vertex then we can color (a, b) by $x(b)$. We also note that there is at most one such vertex.
6. We continue choosing the vertex $v_{i_m} \in N_G(a)$ such that $\phi((a, v_{i_{m-1}})) = x(v_{i_m})$
7. We now proceed to construct the path by starting at a selecting the edge (a, v_{i_m}) then the edge $e_1 = (v_{i_m}, u_1)$ such that $\phi(e_1) = x(a)$ and continue wiht edges with colors alternating between $x(v_{i_m})$ and $x(a)$.

8. Since ϕ is a proper edge coloring, this is a **path** and it must end in a vertex at which one of the two colors is missing.

9. We change the colors of the edges (a, v_{i_k}) to $x_{i_{k-1}}$ and change the colors along the path to obtain the desired edge coloring by $\delta(G) + 1$ colors.