Discrete Optimization
Lecture-14

Ngày 22 tháng 11 năm 2011

Graph COLORING

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In how many colors can you color them so that two provinces sharing borders will have distinct colors?

A brief history

In 1854 Francis Guithrie noticed that he could color the map of England using only four colors so that two counties sharing a border were assigned different colors. He wondered whether every map had this property. He asked his brother who relayed the question to his professor. And so was born the famous four color problem which was not resolved until 1976.

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Graph coloring has many applications. We shall see some applications, prove some basic properties, visit some algorithms with performance guarantees and get an understanding why graph coloring is a hard computational problem.

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- The smallest number of colors needed to color the edges so that two edges sharing a vertex will have distinct colors.

Applications examples

1. The following generic situation is common in many applications. Assume that you have k jobs and n processors that can process the jobs (printers connected to a network, cranes in construction sites, lecture rooms at a university etc.). Some jobs cannot use the same processor. We can construct a graph whose vertices are connected by an edge if they cannot be processed by the same processor.

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Answer

Surprisingly, this is a very hard computational problem! It is indeed so hard that a safe security password scheme can be based on it.

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Imagine that you have a magic box. Inside it lives a very smart machine. It accepts graphs G with n vertices and within $p(n)$ nano-seconds it will decide whether the graph can be properly colored by 3 colors.

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This means that we can also use the 3*-coloring magic box to do the same thing!*

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Given a 3-SAT instance with n variables and m clauses we can construct a graph G with $< 8m$ vertices such that G is 3-colorable if and only if the given $3-SAT$ instance is satisfiable.

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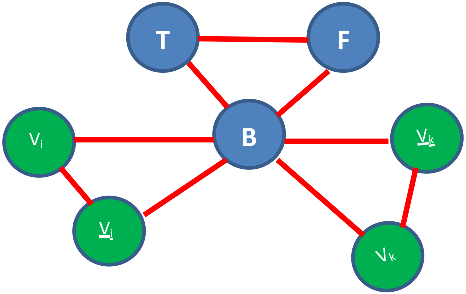
Step 2: With each boolean variable x_i we will associate two vertices labeled $v_i, \underline{v}_i$. For each i, $\{v_i, \underline{v}_i, B\}$ will also form a triangle.


T
F
B
Vi
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Step 3: For each clause we add a subgraph consisting of six new vertices and the edges as described by the following figure: (see next slide).

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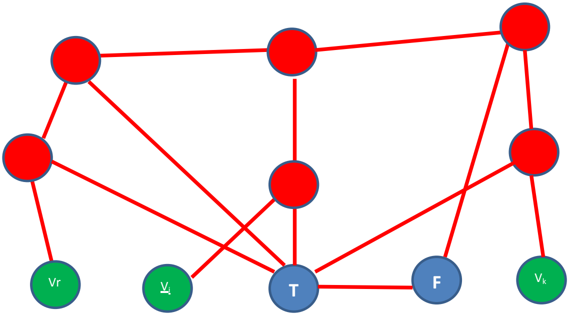
Step 5: for instance, if in the clause $(x_i \vee \bar{x}_j \vee x_k)$ all boolean variables are F then the associated graph G will not be 3 colorable. □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Yes, just try it on a graph with 1000 vertices...

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Watch the blackboard. □