# Discrete Optimization

Ngày 1 tháng 8 năm 2011

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**(The Assignment Problem)**

*n companies submit bids to perform n projects. You wish to assign each job to a different company. How can you do it so that the total cost is minimized?*

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- The Assignment Problem
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**(The Assignment Problem)**

*n companies submit bids to perform n projects. You wish to assign each job to a different company. How can you do it so that the total cost is minimized?*

**(The Traveling Salesman Problem)**

*A traveling salesman needs to visit n cities. He asks his travel agent to design a tour of the cities so that each city will be visited once.*
*We need to help the travel agent to minimize the cost of the tour.*

# An Assignment Problem example

| | C1 | C2 | C3 | C4 | C5 | C6 | C7 | C8 | C9 |
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Bång: Nine companies' bids for nine jobs.

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This is NOT a feasible solution!

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This is a feasible solution! Every company is assigned a job.

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Bảng: A feasible assignment (solution).

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Can you tell whether it is optimal?

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We know how to generate all permutations. There are "only" $9! = 362880$ permutations, no problem even for our laptops.

**Remark**

*Note that this problem has exactly* $9!$ *feasible solutions!*

# The traveling salesman problem

Similarly, the cost of traveling between cities can be represented by a table such as:

| | HAN | HMC | LAX | SFO | SEA | DAD | BKK | PNH |
|---|---|---|---|---|---|---|---|---|
| HAN | 0 | 112 | 1140 | 1216 | 1270 | 174 | 210 | 235 |
| HMC | 112 | 0 | 1004 | 1235 | 1185 | 160 | 190 | 195 |
| LAX | 1140 | 1005 | 0 | 200 | 190 | 1590 | 1210 | 1235 |
| SFO | 1216 | 1133 | 1171 | 0 | 180 | 1275 | 1200 | 1400 |
| SEA | 1270 | 1165 | 140 | 235 | 0 | 1295 | 1220 | 1390 |
| DAD | 170 | 139 | 1600 | 1215 | 1199 | 0 | 189 | 210 |
| BKK | 175 | 113 | 1044 | 1290 | 1286 | 170 | 0 | 205 |
| PNH | 195 | 173 | 1361 | 1485 | 1479 | 179 | 210 | 0 |

Bảng: An instance of the Traveling Salesman Problem.

# A “simple” solution

Assume that the traveling salesman starts in Hanoi and wishes to return to Hanoi. A tour might be the following sequence:

$HAN \rightarrow LAX \rightarrow SEA \rightarrow SFO \rightarrow DAD \rightarrow BKK \rightarrow HMC \rightarrow PNH \rightarrow HAN$.

This tour can be represented by the following 7-permutation $[3, 5, 6, 4, 7, 2, 8]$.

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This tour can be represented by the following 7-permutation $[3, 5, 6, 4, 7, 2, 8]$.

So a simple solution would be to generate all tours (7-permutations of $\{2, 3, 4, 5, 6, 7, 8\}$), for each tour (permutation) calculate its cost and select the cheapest.

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- Find the optimal solution.

And we are left with one problem:

How to **prove** that the solution we found is indeed optimal.

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## (SUMMARY)

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- *The assignment problem and the traveling salesman problem are conceptually very simple and have a very simple solution.*
- *But the simple solution is not practical.*
- *This is typical of Discrete Optimization problems.*

# Key Concepts

- Objective function

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