position, and the same holds for the right cosets of K in L. Hence the set

$$T = \{t \mid t \in L \text{ and } t \text{ takes } \overrightarrow{AB} \text{ to a parallel position}\}$$

is a common set of representatives for the right cosets of H and the right cosets of K in L. Now if X is a set of representatives for the right cosets of L in G, $\{tx \mid t \in T \text{ and } x \in X\}$ is the desired set.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada.

HOW TO CUT ALL EDGES OF A POLYTOPE?

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In the sequel we shall discuss a family of mutually related problems which are somewhat remarkable from two points of view: On the one hand, despite their intuitive appeal and accessibility, the questions seem to have been considered in the literature only in very few isolated instances. On the other hand, the ramifications of the topic reach from pattern recognition (which motivated the investigation of

O'Neil [7]) through the theories of graphs and of convex polytopes to functional analysis (which provided the motivation for Klee [6]).

Let P be a d-polytope (that is, a convex polytope of dimension d in Euclidean d-space E^d; for terminology and results concerning polytopes see [2]); a **cut** of P is any set of edges of P which may be simultaneously intersected by a $(d-1)$-dimensional hyperplane that misses all the vertices of P. We define the **cut-number** $m(P)$ of P as the minimal number of cuts needed to cover all the edges of P. For a trivial example we may take $d = 2$ and P any n-gon; then clearly $m(P) = [\frac{1}{2}(n+1)]$, where $[x]$ is the largest integer not exceeding x.

If $k \leqq d$ then k hyperplanes divide E^d into at most 2^k regions. Therefore, denoting by T^d any d-dimensional simplex, it is obvious that $m(T^d) = \,]\log_2(d+1)[$ where $]x[$ denotes the smallest integer not less than x. This may be generalized to the following conjecture which, although trivial for $d = 2,3$, is open for all $d \geqq 4$.

CONJECTURE 1. *The cut-number of every d-polytope P satisfies*

$$m(P) \geqq m(T^d) = \,]\log_2(d+1)[\,.$$

If C^d denotes the regular d-dimensional cube, it is easily checked that $m(C^3) = 3$ and $m(C^d) \leqq d$. Recently O'Neil [7] proved that any cut of C^d contains at most

$$[\tfrac{1}{2}(d+1)] \binom{d}{[\frac{1}{2}d]}$$

edges; therefore

$$m(C^d) \geqq d2^{d-1}/[\tfrac{1}{2}(d+1)] \binom{d}{[\frac{1}{2}d]}$$

which, using Stirling's formula, leads to a bound of the type $m(C^d) \geqq an^{\frac{1}{2}}$ for a suitable constant $a > 0$. It may be verified that $m(C^4) = 4$, but the reasonably seeming conjecture $m(C^d) = d$ is reported by O'Neil to be false, an example attributed to Paterson implying $m(C^6) \leqq 5$.

This leads to

Problem 1. Determine $m(C^d)$ for $d \geqq 5$.

Denoting by Q^d the (regular) d-dimensional cross-polytope, it is easy to verify that $m(Q^d) \leqq 1 + \,]\log_2 d[$, and it may be conjectured that equality holds for all d. Moreover, we make

CONJECTURE 2. *The cut-number of every centrally symmetric d-polytope P satisfies* $m(P) \geqq 1 + \,]\log_2 d[$.

At least for $d = 3$ Conjecture 2 is true and we have (see Grünbaum [3]):

For every centrally symmetric 3*-polytope P we have* $m(P) \geqq 3$.

As an analogue of Conjecture 1 for simple polytopes we make:

CONJECTURE 3. *If P is a simple d-polytope then* $m(P) \geqq m(C^d)$.

We also venture

CONJECTURE 4. *If P and P′ are isomorphic d-polytopes then*

$$m(P) = m(P').$$

This conjecture appears almost preposterous, in view of the fact that the maximal number of edges in a cut may differ for isomorphic polytopes. For example, any cut of the regular octahedron has at most 6 edges, while the xy-plane determines an 8-edge cut in the isomorphic polytope with vertices $(\pm 1, 0, \frac{1}{4})$, $(0, \pm 1, -\frac{1}{4})$, $(0, 0, \pm 1)$. The chances of the conjecture being true are naturally better if only simple d-polytopes are considered, or if $d = 3$—but even if both conditions are satisfied the proof seems to be very elusive. In this context we make:

CONJECTURE 5. *The maximal number of edges in cuts of every polytope isomorphic to the d-cube* C^d *is the same as for cuts of* C^d *itself.*

If true this conjecture is of interest since polytopes isomorphic to C^d allow cuts of many types not possible with C^d. For example, the convex hull of the 8 points $(\pm 3, \pm 1, 1)$ and $(\pm 1, \pm 3, -1)$ in E^3 is isomorphic to the cube C^3; the plane $x = y$ intersects it in six edges that correspond to the heavily drawn edges of C^3 in Figure 1—but the only 6-edge cuts of C^3 form a circuit of length 6.

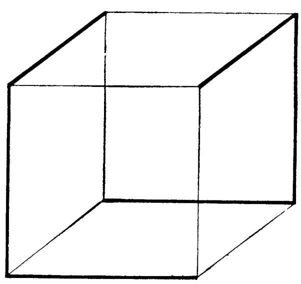

FIG. 1.

Let us call an ***i*-cut** of P ("i" for "isomorphism") a set of edges of P which correspond to the edges of a cut in some polytope P' isomorphic to P, and let $m_i(P)$ be the least number of i-cuts needed to cover all edges of P. Clearly $m_i(P) \leqq m(P)$, and strict inequality holds even for C^3. Indeed $m_i(C^3) = 2$, since the heavy lines in Figure 1 form one i-cut of C^3, while the thin ones form another. Clearly, $m_i(T^d) = m(T^d)$ for all d; we have

Problem 2. Determine $m_i(C^d)$ for $d \geqq 4$.

CONJECTURE 6. *For every d-polytope P,* $m_i(P) \geqq m_i(T^d)$*; moreover, if P has a center of symmetry then* $m_i(P) \geqq m_i(C^d)$.

Instead of d-polytopes it is possible to consider **tessellations** of the $(d-1)$-sphere S^{d-1}, that is, cell-complex decompositions of the unit sphere S^{d-1}, with convex cells. While each d-polytope leads by radial projection to such tessellations, it is well known (see Supnick [**9**], Shephard [**8**]) that not every tessellation is obtainable as such a projection. If the cut-number $m(Q)$ of a tessellation Q of S^{d-1} is defined as the least number of great $(d-2)$-spheres (which miss all vertices of Q) needed to intersect all edges of Q, one may reformulate for tessellations Conjecture 1. However, the analogue of Conjecture 2 has to be modified since (see Grünbaum [**3**]) there exist centrally symmetric tessellations Q of the 2-sphere with $m(Q) = 2 < m(C^3) = 3$. This leads to:

Problem 3. Determine $\min_Q\{m(Q)\}$, where Q ranges over all centrally symmetric tessellations of the $(d-1)$-sphere.

We call a t-**cut** ("t" for "topological") of a d-polytope P any set of edges intersectable by a suitable homeomorphic image S of a $(d-2)$-sphere S^{d-2} in boundary of P, provided $F \cap S$ is a topological j-cell, $j \leqq k-1$, for every k-face F of P. We define $m_t(P)$ as the least number of t-cuts needed to cover all edges of P. Clearly $m_t(P) \leqq m_i(P)$, and we make

CONJECTURE 7. *For every* $d \geqq 4$ *there exists a d-polytope* P_d *such that* $m_t(P_d) < m_i(P_d)$.

In some contrast to that conjecture is the following fact:

For every 3-polytope P we have $m_i(P) = m_t(P)$.

In order to prove this assertion it is clearly enough to show that every t-cut of a 3-polytope is also an i-cut. Let P be a 3-polytope, let H be a t-cut of P, and let P^* be a polytope dual to P. Then there is a natural one-to-one correspondence between the edges of P and those of P^*, such that to edges of each t-cut H of P there correspond the edges of a simple circuit H^* in P^*, and vice versa. According to a beautiful theorem of Barnette [**1**], there exists a 3-polytope $\bar{P}$ isomorphic to P^* such that the circuit $\bar{H}$ which corresponds to H^* is the (sharp) **shadow boundary** of $\bar{P}$ for projection (illumination) in the direction of a suitable line L. Denoting by P' a polytope polar to $\bar{P}$ with respect to a sphere centered at an interior point of $\bar{P}$, and by H' the t-cut of P' that corresponds to the circuit $\bar{H}$ of $\bar{P}$, it follows that the plane through the origin perpendicular to L intersects all the edges in H'; hence H' is a cut, and our assertion is proved.

Because of the duality between the t-cuts of 3-polytopes and simple circuits on the dual polytopes, it is possible to deduce some properties of $m_t(P)$ from known results on simple circuits (see [**4**] for a survey of results and for references). As an example we mention the following fact:

There exist 3-polytopes P with arbitrarily many edges, having all vertices of valence $\leqq 6$ and all faces with at most 6 sides, such that $m_t(P) \geqq (e(P))^b$, where b is a positive constant (we may even take $b \geqq 1-(\log 8/\log 13) \approx 0.19$), and $e(P)$ is the number of edges of P.

We could not decide:

Problem 4. Does there exist a constant b such that $m(P) \leqq b$ (or at least $m_t(P) \leqq b$) for every 3-polytope P having only vertices of valence at most 4 and faces with at most 4 sides?

The ideas discussed above may be modified to apply to graphs. Let G be a graph; a ***g*-cut** of G (cocircuit; cocyle in Harary [**5**]) is a set of edges of G which separates G and is minimal with respect to that property (i.e., no proper subset separates). We define $m_g(G)$ to be the least number of g-cuts needed to cover all edges of G. Clearly, if G is the graph of a d-polytope P, then every t-cut of P is a g-cut of G. In case $d=3$ the converse is also true, but for $d \geqq 4$ it is not known whether every g-cut of the graph of a d-polytope is a t-cut of the polytope. Analogues of Conjectures 1 and 2 may be formulated for g-cuts of the graphs of d-polytopes.

The properties of graphs of d-polytopes lead naturally to some problems concerning g-cuts of graphs.

Problem 5. Determine $m_g(k)$, the least value of $m_g(G)$ when G varies over all k-connected graphs.

It is not hard to verify, using the graphs of the tetrahedron, octahedron, and icosahedron, that $m_g(3) = m_g(4) = m_g(5) = 2$, but the value of $m_g(6)$ is not known. However, we make

CONJECTURE 8. *If G is a k-connected graph and if G contains a subgraph isomorphic to a subdivision of the complete graph with $k+1$ nodes, then $m_g(G) \geqq \log_2(k+1)$.*

The various problems posed above may also be generalized in a completely different direction. Let P be a d-polytope, and let k be an integer with $1 \leqq k \leqq d-2$. We call a ***(k)-cut*** of P any set of k-faces of P which may be simultaneously intersected by a suitable $(d-k)$-flat H such that $H \cap F = \emptyset$ for all faces F of P of dimension less than k. The definition of (k)-cut-number of P, etc., is obvious. The cuts we discussed above correspond to $k=1$. Unfortunately, no non-trivial results on those notions seem to be known for $k>1$, except that a result of Klee [**6**] may be interpreted as follows:

Every centrally symmetric d-polytope has a $(d-2)$-cut comprising at least $2d$ $(d-2)$-faces.

Added in proof: The validity of Conjecture 1 has been established by David W. Barnette; his paper "Cut numbers of convex polytopes" will appear in the journal *Geometriae Dedicata.*

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CORRECTIONS TO "THE HADAMARD MAXIMUM DETERMINANT PROBLEM"

(This MONTHLY, 79(1972) 626–630.)

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Please note the following:

1. The correct address of J. L. Brenner is: 10 Phillips Rd. Palo Alto, CA 94303.
2. The research was supported by NSF GP 32527.

CLASSROOM NOTES

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A UNIFIED PROOF OF SEVERAL BASIC THEOREMS OF REAL ANALYSIS

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1. Introduction. The place of continuity in elementary real analysis is justified by its role as a hypothesis in three important theorems. Specifically, if f is a continuous real-valued function on a closed bounded interval $[a, b]$, then

(i) f is bounded on $[a, b]$ (and actually attains maximum and minimum values);

(ii) f has the intermediate value property on $[a, b]$; and

(iii) f is Riemann integrable on $[a, b]$.

It is the purpose of this note to present proofs of these theorems in which the part played by continuity is isolated and shown to enter into each proof in essentially the same way; in effect, the three theorems are derived as corollaries of a single lemma.