A Problem in Graph Coloring

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RESEARCH PROBLEMS

EDITED BY VICTOR KLEE

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Matcrial should be sent to Victor Klee, Department of Mathematics, University of Washington, Seattle, WA 98105.

A PROBLEM IN GRAPH COLORING

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A graph is said to be *colorable by k colors* if its set of nodes can be partitioned into k disjoint subsets such that no edges connect nodes in the same subset, and is said to be *k-chromatic* if it is colorable by k colors but not by $k-1$ colors. In recent years considerable attention has been devoted to the following problem: Given positive integers k and n, with $k \geqq 3$ and $n \geqq 3$, do there exist graphs which are k-chromatic while containing no circuits of length $\leqq n$?

The following results have been obtained: Zykov [1949] and Mycielski [1955] proved the existence of k-chromatic graphs of arbitrarily large k having no circuits of length $\leqq n=3$. Descartes [1954] and Kelly-Kelly [1954] obtained the same result for $n=5$, and Nešetřil [1966] for $n=7$. Erdös [1959] established by probabilistic methods the existence for arbitrary k and n, while Lovász [1968] gave direct constructions for such graphs. The graphs constructed in all those papers are "very large" and, in particular, contain nodes of valence large in comparison to k.

It is of some interest to note that the existence of k-chromatic graphs with no circuits of length $\leqq n$ was recently (Taylor [1969]) used to answer in the negative a question in model theory posed by Mycielski [1964].

On the other hand, a result of Brooks [1941] asserts: *If all nodes of a connected graph G have valence at most k, where $k \geqq 3$, then either G is k-colorable or else G is the complete graph with $k+1$ nodes.*

Some years ago (see Erdös [1964]) observations like the above led me to formulate the following conjecture:

CONJECTURE. *If $k \geqq 3$ and $n \geqq 3$ are integers, there exist k-chromatic, k-valent graphs $G(k, n)$ that contain no circuits of length $\leqq n$.*

The progress in settling the conjecture seems to have been very slow. Graphs $G(3, n)$ are easy to construct, for example from circuits with an odd number of edges and from 3-valent graphs with girth $\geqq n$ (concerning such graphs see, for example, Sachs [1964]). Recently Chvátal [1970] constructed a 12-node graph $G(4, 3)$. In the following lines I shall describe a 25-node graph $G=G(4, 4)$ I found in 1963. As far as I know, no other cases of the conjecture have been decided.

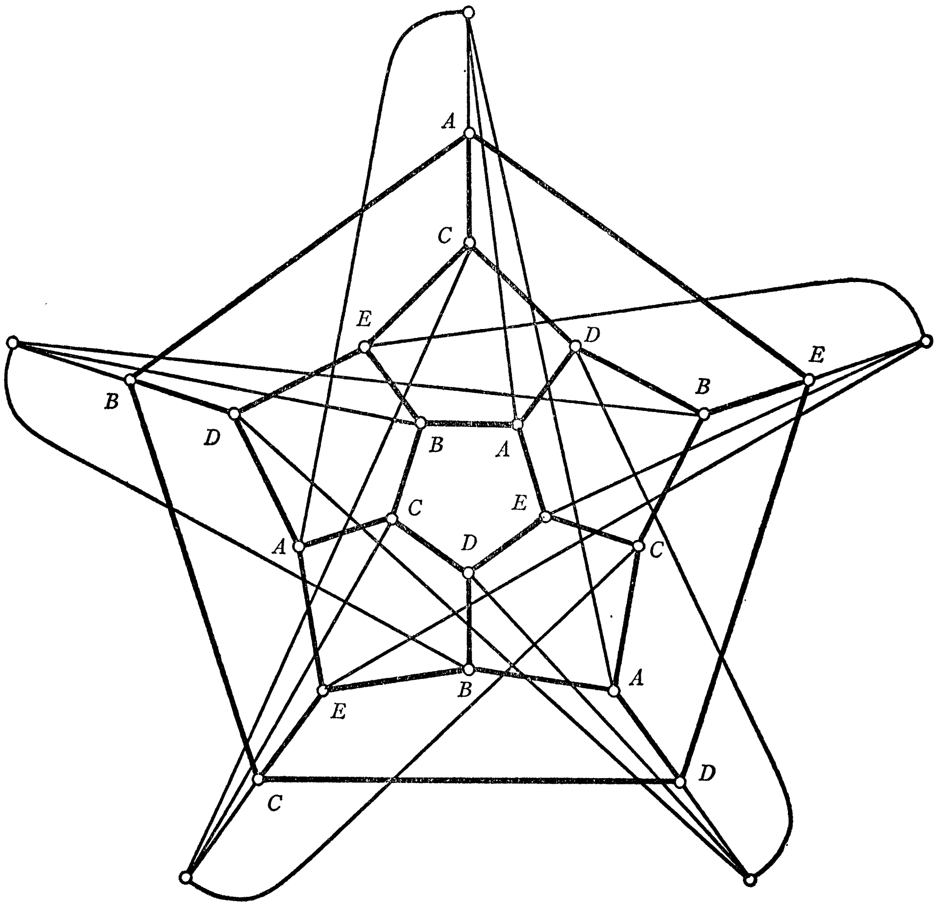

FIG. 1

The graph G (see Figure 1) may be described as follows: Out of the 25 nodes of G, 20 nodes and the edges connecting them form the net of a regular dodecahedron (heavy edges in Figure 1). It is well known that there are 10 quadruples of vertices of the dodecahedron which are the vertices of a regular tetrahedron. These ten quadruples fall into two sets of five quadruples each, the quadruples of each set covering simply the vertices of the dodecahedron. One such set is represented in the heavily drawn part of G by nodes bearing the same designation. The graph G is obtained by taking five additional nodes (the "outer" ones in Figure 1) and joining each of them to the four nodes in one of the quadruples. Thus constructed, G is obviously 4-valent, and contains no circuit with less than 5 edges. It is also easily checked that the action of the group of automorphisms of G on the five outer nodes coincides with that of the alternating group.

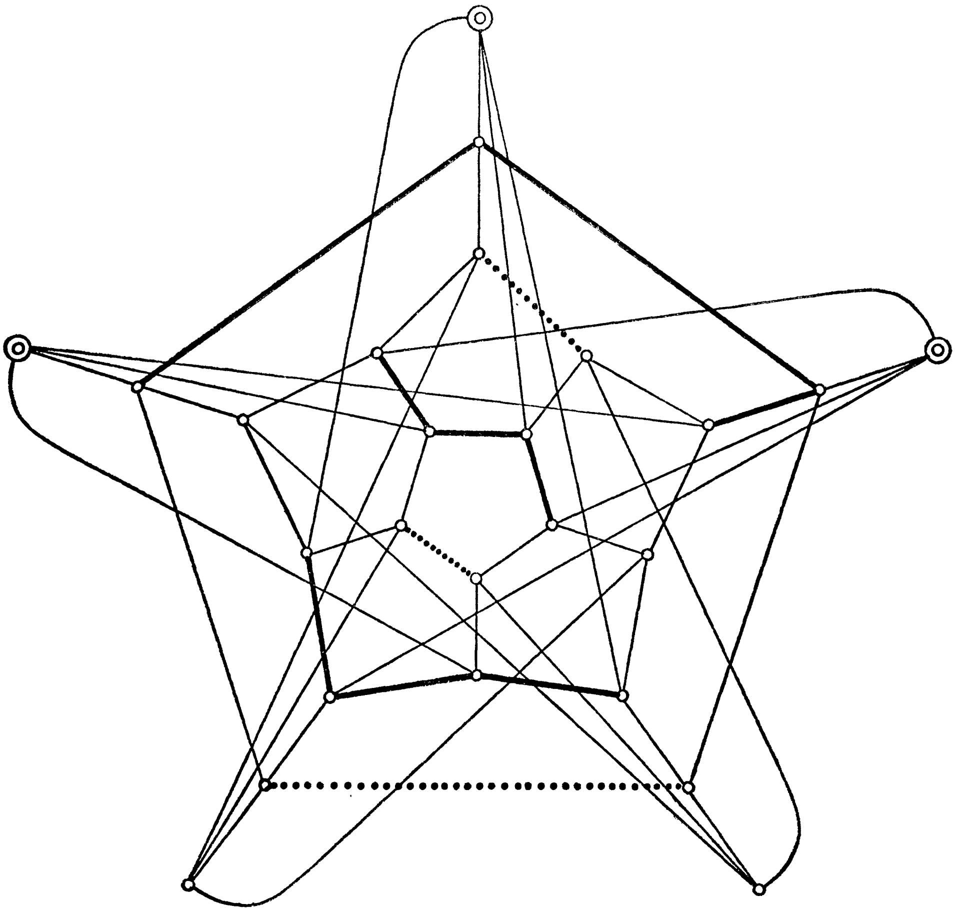

FIG. 2

In order to show that G is not 3-colorable, assume that a 3-coloring of G is given; then there are two possibilities to be considered for the outer five nodes:

(i) some three of them have the same color;

(ii) two nodes have one color, two another, the last node having the third color.

In case (i) we may, without loss of generality, assume that the three nodes in question are those marked by two circles in Figure 2; then the nodes in the paths indicated by heavy edges are colored by the remaining two colors. But then for at least one of the edges indicated by dotted lines, both nodes incident to the edge must have the same color, that of the outer three nodes.

In case (ii) we may assume, again without loss of generality, that the outer nodes are colored by 1, 2, 3 as indicated in Figure 3. Then, if the node indicated by two circles is colored 2, we arrive at a contradiction by following the arrow

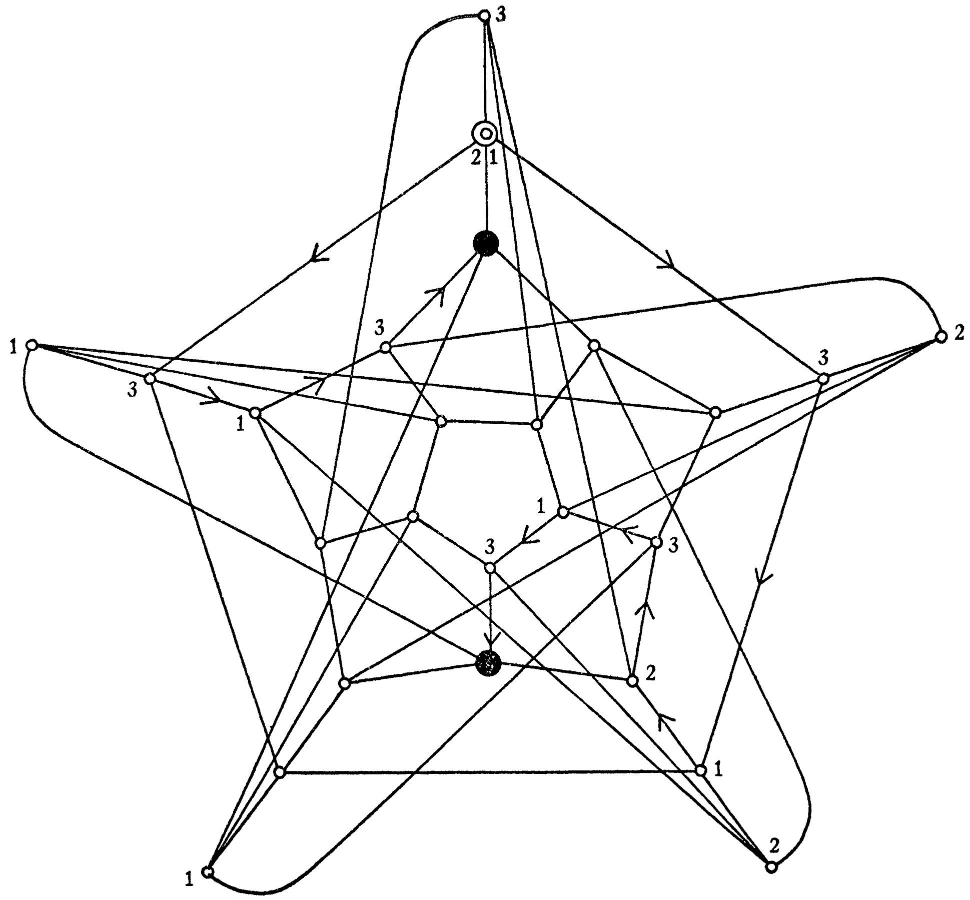

FIG. 3

leading from it to the left, there being no color available for the node indicated by the black disc. If, on the other hand, the two-circle node is colored 1 we reach a contradiction by following the arrow to the right.

Thus G is not 3-colorable, and we established that it is a graph of type $G(4, 4)$.

The author is indebted to Professor R. A. Duke for many helpful suggestions.

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CLASSROOM NOTES

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ON THE SIMILARITY OF PARTIALLY ORDERED SETS

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It is well known and easy to prove [1] that for linearly ordered sets $(R, \leqq)$ and $(S, \leqq')$, if there exists a one-to-one mapping p from R onto S which preserves order (i.e., $x \leqq y$ implies $p(x) \leqq' p(y)$), then $(R, \leqq)$ is similar to $(S, \leqq')$ since the converse mapping p_{-1} of p also preserves order.

It is a curious fact that in the case of partially ordered sets $(P, \leqq)$ and $(Q, \leqq')$, even a stronger hypothesis does not imply that $(P, \leqq)$ is similar to $(Q, \leqq')$. For instance, as shown in the example below (suggested by Daryl R. Fischer), if there exists a one-to-one mapping f from P *onto* Q such that f preserves order and if there exists a one-to-one mapping g from Q *onto* P such that g also preserves order, then it is not necessary that P be similar to Q, i.e., it is not necessary that there exist a one-to-one mapping h from P onto Q such that both h and its converse h_{-1} preserve order.

Example 1. Let $(P, \leqq)$ and $(Q, \leqq')$ be partially ordered sets represented in the diagram below, where $p_i < p_j$ if and only if there exists an upward connecting line from p_i to p_j. Similarly, $q_i <' q_j$ if and only if there exists an upward connecting line from q_i to q_j. Otherwise, the distinct elements are not comparable.