A MEASURE OF ASYMMETRY FOR PLANE CONVEX SETS†

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1. *Introduction.*

For any plane convex body‡ K we consider partitions of K by straight lines L_1, L_2, L_3, subject to the condition§ (see Fig. 1):

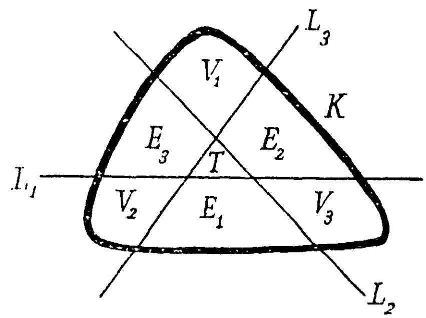

Fig. 1

$$E_i \leqslant V_i \text{ for } i = 1, 2, 3. \quad (*)$$

Let
$$f(K;\, L_1, L_2, L_3) = \frac{T}{E_1+E_2+E_3}.$$

We are interested in the functional $f(K)$ defined by

$$f(K) = \sup \{f(K;\, L_1, L_2, L_3) \mid L_1, L_2, L_3 \text{ satisfy } (*)\},$$

and we shall prove the following

THEOREM. *For every plane convex body K*

(i) $0 \leqslant f(K) \leqslant \frac{1}{24}$;

(ii) $f(K) = 0$ *if and only if K has a centre of symmetry*;

(iii) $f(K) = \frac{1}{24}$ *if and only if K is a triangle.*

Thus $f(K)$ is a measure of asymmetry|| for plane convex bodies; it is obviously an affine-invariant measure of asymmetry.

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‡ A convex body is a compact convex set with non-empty interior.

§ We shall denote a convex set and its area by the same letter.

|| See [5] for a summary of results on measures of asymmetry and references.

Our estimate $f(K) \leqslant \frac{1}{24}$ generalizes the well-known result of Sholander [6]† that $f(K; L_1, L_2, L_3)$ is at most $\frac{1}{24}$ if, in addition to (*), the relation $E_1 = E_2 = E_3$ is assumed.

In §2 we shall prove the assertion (i) of the theorem; as a by-product of the proof we obtain (iii). Assertion (ii) shall be proved in §3, while §4 contains some remarks and problems.

2. *Proof of Assertion* (i).

Since $f(K)$ is obviously non-negative‡, we shall prove only $f(K) \leqslant \frac{1}{24}$. We begin by remarking that, by standard compactness arguments, for any given K the functional $f(K; L_1, L_2, L_3)$ assumes a maximal value for certain lines L_1, L_2, L_3, and also that $f(K)$ assumes its maximal value for a certain convex body K^*. Using simple geometric arguments, we shall first determine K^* and some properties of the L_i's which maximize $f(K; L_1, L_2, L_3)$. The analytic determination of the L_i's and of $f(K)^*$ will complete the proof of (i) and (iii).

For any given K a necessary condition for the maximum of $f(K; L_1, L_2, L_3)$ is (see Fig. 2) that the segment C_1A_2 have the same length as A_3B_1, and similarly $C_2A_3 = A_1B_2$ and $C_2A_1 = A_2B_3$. Indeed, if *e.g.* $C_1A_2 > A_3B_1$, then for a suitable line L_1^* through the midpoint A_1^* of A_2A_3 we would have $f(K; L_1^*, L_2, L_3) > f(K; L_1. L_2, L_3)$.

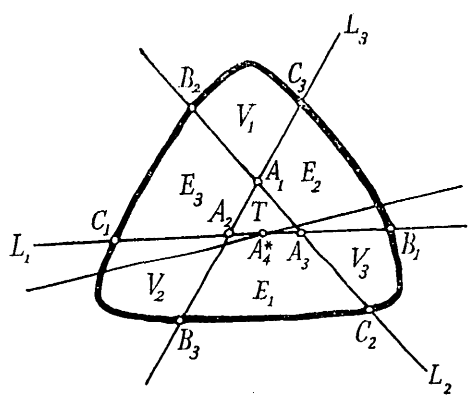


Fig. 2

Let now any K, L_1, L_2, L_3 be given and let K^* be the triangle determined by the straight lines B_1C_3, B_2C_1, and B_3C_1 (*cf.* Fig. 2). Obviously the lines L_i satisfy condition (*) with respect to K^*, and

$$f(K^*; L_1, L_2, L_3) \geqslant f(K; L_1, L_2, L_3).$$

† Conjectured by R. C. Buck and E. F. Buck [**1**]; proved also by H. G. Eggleston [**2**] (this proof is reproduced in Eggleston's books [**3**, **4**]).

‡ The existence of at least one set of lines L_1, L_2, L_3, satisfying (*) is a consequence of the existence of sixpartite points (Buck and Buck [**1**]; sixpartite points correspond to the case $T = 0$, $E_i = V_j$ for $i, j = 1, 2, 3$.

Equality holds here if and only if the boundary of K coincides with that of K^* in $E_1 \cup E_2 \cup E_3$. It follows that the maximum of $f(K)$ is assumed for K a triangle.

For a triangle K with vertices D_1, D_2, D_3, the maximum of $f(K; L_1, L_2, L_3)$ can be achieved only if $V_i = E_i$ for all i. Indeed, if *e.g.* $V_1 > E_1$ then for a suitable triangle K^* with vertices D_1^*, D_2^*, D_3^* (see Fig. 3) we would have $f(K^*; L_1, L_2, L_3) > f(K; L_1, L_2, L_3)$.

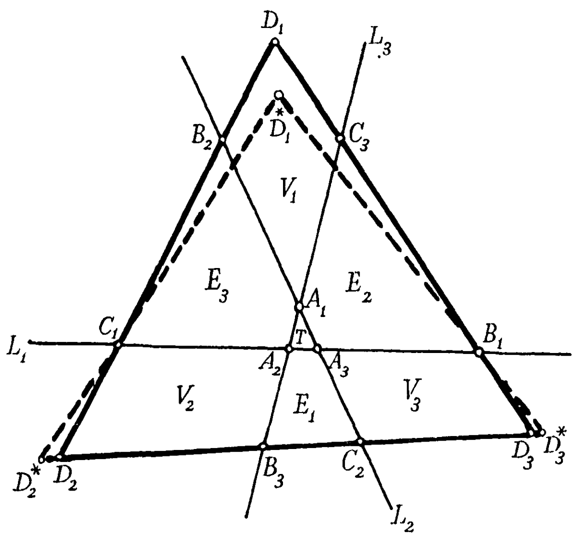


Fig. 3

According to the above, the Assertion (i) of the theorem shall be proved if we show that for all lines L_1, L_2, L_3, such that (using the notations of Fig. 3) $E_i = V_i$ for $i = 1, 2, 3$, and

$$C_1A_2 = A_3B_1,\quad C_2A_3 = A_1B_2,\quad C_3A_1 = A_2B_3, \tag{**}$$

the inequality $24T \leqslant E_1 + E_2 + E_3$ holds. We find it convenient to prove the stronger statement, viz. $8T \leqslant E_1$ provided (**) holds and $V_1 = E_1 \leqslant E_j$, $j = 2, 3$, with no assumption on V_2 and V_3. For simplicity of computation we take T (and L_1, L_2, L_3) fixed as indicated in Fig. 4 and proceed as follows (assuming, without loss of generality, that $a \geqslant b$).

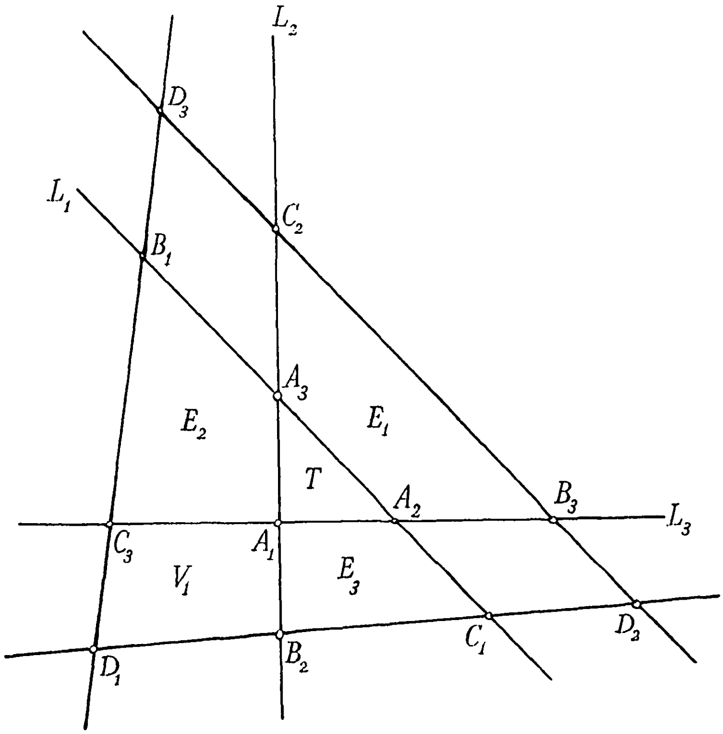
First we find

$$x_0 = -\frac{1+c}{(1+c)^2 - (a-c)(b-c)}[a(1+c)+b(a-c)]$$

$$y_0 = -\frac{1+c}{(1+c)^2 - (a-c)(b-c)}[b(1+c)+a(b-c)].$$

From $E_1 = V_1$ it follows that

$$(a-b)^2c^2 + 2[a(1+a)+b(1+b)]c - [ab(2+a+b)+(a+b+ab)(ab-1)] = 0; \tag{***}$$



$A_1 = (0, 0)$ $B_1 = (-c, 1+c)$ $C_1 = (1+c, -c)$
$A_2 = (1, 0)$ $B_2 = (0, -b)$ $C_2 = (0, 1+b)$
$A_3 = (0, 1)$ $B_3 = (1+a, 0)$ $C_3 = (-a, 0)$
$D_1 = (x_0, y_0)$

Fig. 4

similarly, $E_3 \geqslant E_1$ implies $b(1+c)+c \geqslant a+b+ab$ and therefore $c \geqslant a$. Combining this inequality with (***) and simplifying, it follows that

$$a^3+a^2+a \leqslant b(2a^2-1); \qquad (****)$$

together with $b \leqslant a$ this implies $a^3-a^2-2a \geqslant 0$. Since $a > 0$ we obtain $a \geqslant 2$.

Now, as easily checked,

$$\frac{a^3+a^2+a}{2a^2-1} \geqslant 2 \text{ for } a \geqslant 2,$$

and therefore (****) yields $a \geqslant b \geqslant 2$. But then $E_1 \geqslant 4 = 8T$, as claimed in (i). Equality holds if and only if $a=b=2$; then $c=2$, and $E_i = V_i = 4$, $T = \frac{1}{2}$. This establishes (iii).

3. *Proof of Assertion* (ii).

It is well known that a plane convex body K is centrally symmetric if and only if all the straight lines which bisect the area of K are concurrent (at the centre of K). Therefore, if K is not centrally symmetric there exists three non-concurrent lines L_1, L_2, L_3, each of which bisects the area of K. Obviously $T > 0$, and since the L_i's are area-bisectors of K

we have (in the notation of Fig. 1) $V_i = T + E_i$ for $i = 1, 2, 3$; thus $V_i \geqslant E_i$, condition (*) is fulfilled, and $f(K; L_1, L_2, L_3) > 0$.

There remains to be shown that $f(K) = 0$ for centrally symmetric K. Suppose that, on the contrary, this is not true, *i.e.* that there exists a centrally symmetric K and lines L_1, L_2, L_3 such that $f(K; L_1. L_2, L_3) > 0$. Compactness arguments again establish the existence of extremal K and L_i's.

With regard to the possible positions of the centre O of K relative to L_1, L_2, L_3, it is immediate that O cannot belong to T or to $E_1 \cup E_2 \cup E_3$. Indeed, in the first case (see Fig. 5) each of the lines M_i through O parallel to L_i bisects the area of K, and therefore

$$E_1 + V_2 + V_3 \leqslant V_1 + E_2 + E_3$$
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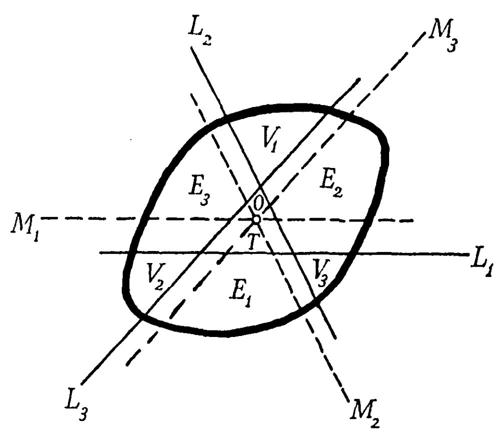

Fig. 5

with strict inequality at least in one of the relations; adding the three inequalities we obtain $V_1 + V_2 + V_3 < E_1 + E_2 + E_3$, in contradiction to (*).

The possibility that O belongs, *e.g.*, to E_1 is at once contradicted by the condition $V_1 \geqslant E_1$ (see Fig. 6).

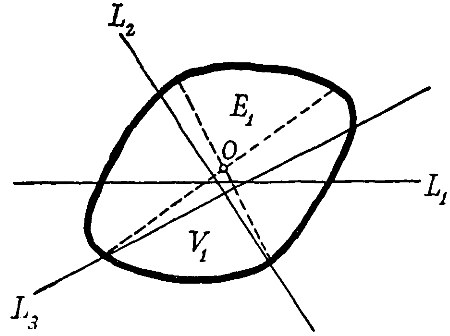

Fig. 6

In order to dispose of the remaining possibility let us assume that O belongs to V_1 (see Fig. 7). Let B_i^*, C_i^*, be the points symmetric to B_i, C_i, with respect to the centre O. Denote by P the parallelogram with vertices P_0, P_1, P_2, P_3 whose sides are determined by the lines $C_2 B_3$, $B_2^* C_3$, $B_3^* C_2^*$, $B_2 C_3^*$. Considering the shaded areas it follows that (*) is satisfied by the lines L_1, L_2, L_3, with respect to P, and that

$$f(K;\, L_1, L_2, L_3) \leqslant f(P;\, L_1, L_2, L_3).$$

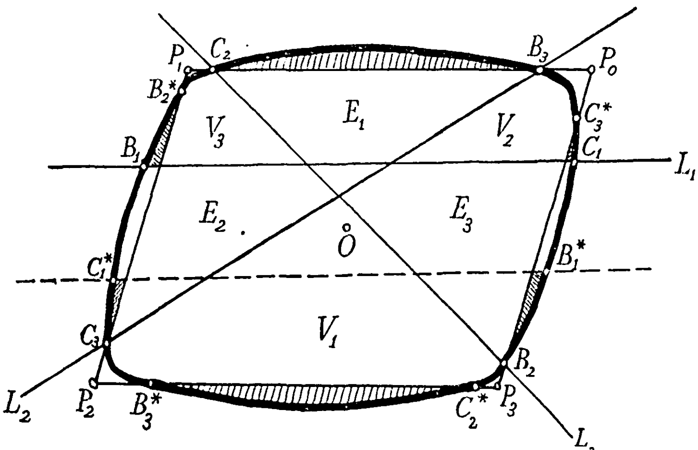

Fig. 7

We may therefore restrict our attention to parallelograms.

Assuming the configuration to yield maximal $f(P, L_1, L_2, L_3)$ we have (as shown in §2) $B_1 A_3 = A_2 C_1$ (see Fig. 8). Substituting for L_1 the line L_1^* determined by the midpoint A_1^* of $A_2 A_3$ and by P_0 (or P_1), it follows that $f(P, L_1, L_2, L_3) \leqslant f(P, L_1^*, L_2, L_3)$. (Note that if L_1, L_2, L_3 satisfy (*), so do L_1^*, L_2, L_3). But if $C_1 = P_0$ (or $B_1 = P_1$), then $C_3 A_1 = A_2 B_3$ obviously contradicts (*). The contradiction reached completes the proof of (ii).

4. *Remarks.*

(a) Using the notations of §1, let

$$g(K;\, L_1, L_2, L_3) = \max\left\{\frac{T}{E_1}, \frac{T}{E_2}, \frac{T}{E_3}\right\},$$

the lines L_i satisfying condition (*). As in §3 it follows that $g(K)$ is a measure of asymmetry; obviously $3f(K) \leqslant g(K)$. Probably $g(K) \leqslant \frac{1}{8}$, but our arguments do not establish this.

(b) Similarly, if $h(K;\, L_1, L_2, L_3) = \dfrac{T}{E_1}$ for L_i satisfying

$$E_1 = E_2 = E_3 \leqslant V_1 = V_2 = V_3,$$

it follows from part (i) of our theorem that $h(K) \leqslant \frac{1}{8}$ with equality only if K is a triangle. Also, $h(K) = 0$ if K is centrally symmetric. One may conjecture that $h(K) = 0$ only for centrally symmetric K, although no proof of this seems to be known.

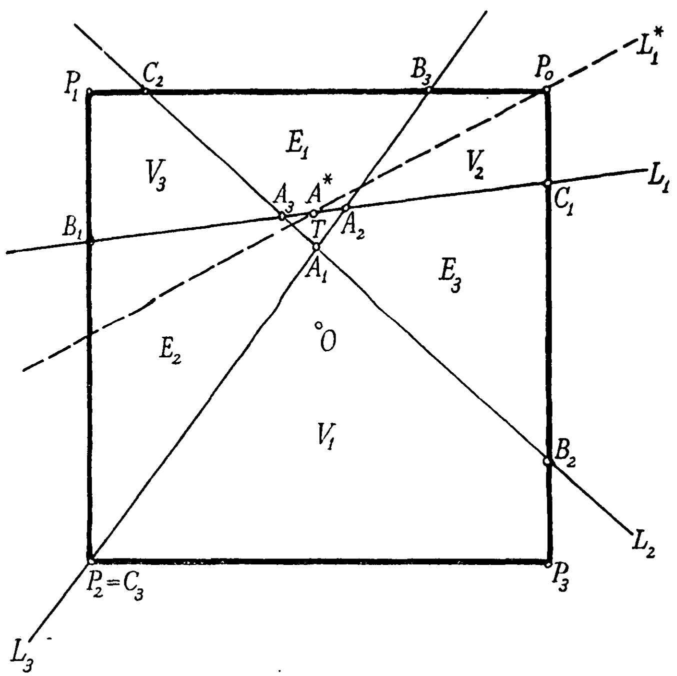

Fig. 8

(c) It would be interesting to investigate the analogues of $f(K)$ in higher dimensions. It seems that one reasonable generalization to E^3 would consist in asking for the maximum of the volume of the central tetrahedron if its bounding planes are supposed to satisfy conditions of the type "all vertex regions have the same volume, and so do all edge regions and vertex regions", and possibly some inequalities of the type (*).

(d) Our theorem obviously implies the following statement: For any convex body K in the plane, and any lines L_1, L_2, L_3, satisfying (*) we have $0 \leqslant T/K \leqslant 49$. Equality on the left holds if and only if K is centrally symmetric, and on the right if and only if K is a triangle. Thus T/K is another measure of asymmetry. It is interesting to note that the direct proof of $T/K \leqslant \frac{1}{49}$ seems to be more complicated than that of our theorem.

References.

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