A GENERALIZATION OF THEOREMS OF KIRSZBRAUN AND MINTY[1]

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The following result of Kirszbraun [2] is well known (see [4] for a simple proof and for references to other relevant papers, [1] for another generalization):

KIRSZBRAUN'S THEOREM. *Let x_i, y_i, $1 \leqq i \leqq m$, and x be points in Euclidean n-space E^n such that $\|x_i - x_j\| \geqq \|y_i - y_j\|$ for $1 \leqq i, j \leqq m$. Then there exists $y \in E^n$ satisfying $\|x_i - x\| \geqq \|y_i - y\|$ for $1 \leqq i \leqq m$.*

Recently, Minty [3] established the following theorem (stated in [3] for inner-product spaces only, but the proof applies equally well in the present formulation):

MINTY'S THEOREM. *Let x_i, $1 \leqq i \leqq m$, be points of a real Hausdorff linear topological space X (or a complex space, provided only real parts of the functionals are considered), let f_i, $1 \leqq i \leqq m$, be continuous linear functionals over X such that $(f_i - f_j)(x_i - x_j) \geqq 0$ for $1 \leqq i, j \leqq m$, and let a point $x \in X$ [resp. a continuous linear functional f] be given. Then there exists a continuous linear functional f [resp. a point $x \in X$] such that $(f_i - f)(x_i - x) \geqq 0$ for $1 \leqq i \leqq m$.*

In the present note we shall prove a theorem which has both Kirszbraun's and Minty's results as immediate corollaries. The proof is analogous to those in [4] and in a preliminary version of [3], for a copy of which the author is indebted to Professor Minty.

THEOREM. *Let a_i, b_i, $1 \leqq i \leqq m$, be points in E^n such that (we denote by $(x; y)$ the inner product of x and y)*

$$(a_i - a_j;\ b_i - b_j) \geqq 0 \qquad \text{for } 1 \leqq i,\ j \leqq m. \tag{1}$$

Let reals α and β be given, $\alpha^2 + \beta^2 > 0$. Then there exists a point $c \in E^n$ such that $(a_i + \alpha c;\ b_i + \beta c) \geqq 0$ for $1 \leqq i \leqq m$.

REMARK. The theorem may fail if $\alpha = \beta = 0$, as shown by the following example: $n = m = 2$, $\|a_1\| = \|a_2\| = \|b_1\| = \|b_2\| = 1$, $(a_1; a_2) = 0$, $b_1 = -a_1$, $b_2 = a_2$.

PROOF OF THE THEOREM. If $\alpha\beta > 0$ the assertion is obvious; therefore we shall consider only the case $\alpha\beta \leqq 0$. Due to the symmetry of

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the statement, and the possibility to replace α, β by $\lambda\alpha$, $\lambda\beta$ for $\lambda\neq 0$, it is sufficient to consider the case $\alpha\geqq 0$, $\beta=-1$. Let the functional F be defined for $x\in E^n$ by $F(x)=\min\{0;\ (a_i+\alpha x;\ b_i-x),\ 1\leqq i\leqq m\}$. Since $F(x)$ is the minimum of a finite number of functionals of a very special form and of degree at most 2 in x, and is nonpositive, $F(x)$ can be easily shown to attain its maximum at some point c. The proof of the Theorem shall be completed if we show that $F(c)=0$. Let us therefore assume, if possible, that $F(c)<0$. It follows that $(a_i+\alpha c;\ b_i-c)<0$ for at least one i. We assume, without loss of generality, that

(2a) $$(a_i+\alpha c;\ b_i-c)<0 \qquad \text{for } 1\leqq i\leqq k, \text{ with } k\geqq 1,$$

(2b) $$(a_i+\alpha c;\ b_i-c)\geqq 0 \qquad \text{for } k\leqq i\leqq m.$$

Now, the origin 0 is in the convex hull K of the set $\{a_i+2\alpha c-\alpha b_i\colon 1\leqq i\leqq k\}$. Indeed, if this were not the case, there would exist a hyperplane strictly separating 0 and K; in other words, there would exist $y\in E^n$ such that

(3) $$0<(y;\ a_i+2\alpha c-\alpha b_i) \qquad \text{for } 1\leqq i\leqq k.$$

Then $F(c-\epsilon y)=\min\{0;(a_i+\alpha c-\epsilon\alpha y;\ b_i-c+\epsilon y),\ 1\leqq i\leqq m\}$ $=\min\{0;\ (a_i+\alpha c;\ b_i-c)+\epsilon(y;\ a_i+2\alpha c-\alpha b_i)-\alpha\epsilon^2(y;\ y),\ 1\leqq i\leqq m\}$ could be made, because of (2) and (3), greater than $F(c)$ for sufficiently small $\epsilon>0$, in contradiction to the choice of c. Thus 0 is in K, and there exist reals γ_i, $1\leqq i\leqq k$, such that $\gamma_i\geqq 0$, $\sum\gamma_i=1$ and $\sum\gamma_i(a_i+2\alpha c-\alpha b_i)=0$. (All summations are from 1 to k.) Denoting $a=\sum\gamma_i a_i$ and $b=\sum\gamma_i b_i$ we have

(4) $$a+2\alpha c-\alpha b=0.$$

On the other hand it follows from (1) that

(5) $$0\leqq\frac{1}{2}\sum_i\sum_j\gamma_i\gamma_j(a_i-a_j;\ b_i-b_j)=\sum\gamma_i(a_i;\ b_i)-(a;\ b).$$

Multiplying the inequalities (2a) by γ_i and adding, there results $0>\sum\gamma_i(a_i+\alpha c;\ b_i-c)=\sum\gamma_i(a_i;\ b_i)-(a;\ c)+a(b;\ c)-\alpha(c;\ c)$.

Combining this inequality with (5) we obtain

$$0>(a;\ b)-(a;\ c)+\alpha(b;\ c)-\alpha(c;\ c)$$

which reduces, because of (4) and $\alpha\geqq 0$, to

$$0>\alpha(b-c;\ b-c)\geqq 0.$$

The contradiction reached proves $F(c)=0$, and thus also the Theorem.

REMARK. In order to derive the results of Kirszbraun and Minty from the above theorem, we first note that no generality is lost in either case by assuming $x=0$. Minty's theorem is then immediately reduced to the case $\alpha=0$, $\beta=-1$, while Kirszbraun's theorem is the case $\alpha=1$, $\beta=-1$, $a_i=x_i+y_i$, $b_i=x_i-y_i$, and $y=-c$.

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