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Deflection constructions

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A simple construction described in [2] and [3] leads to unexpectedly varied outcomes. The basic step of the construction is indicated in Figure 1(a). Given two points M and V, mirror V in M to obtain V'. This can be formalized by V' = 2M – V. The construction in [2] consists in selecting as points M successively the vertices $M_1, M_2, \ldots M_n, M_{n+1} = M_1$ of an n-gon P. Starting from an arbitrary point $V = V_0$, a sequence of points V_j is obtained by applying the basic step with vertex M_j to point V_{j-1} already constructed. As shown in more detail in [2], there are the following possibilities, assuming that P is not reduced to a point:

(i) If n is even, then either the points V_j are the vertices of an n-gon for every choice of V_0, or else these points are equidistant on a ray (half apeirogon). The former happens if and only if P is such that the centroid of the even-labeled M_j's coincides with the centroid of the odd-labeled ones.

(ii) If n is odd, the sequence of V_j's repeats after 2n steps regardless of V_0 and P; however, for a certain choice of V_0, unique for every P, the sequence of V_j repeats already after n steps.

In the present note we shall investigate a generalization of this construction. The basic step is illustrated in Figure 1(b). As before,

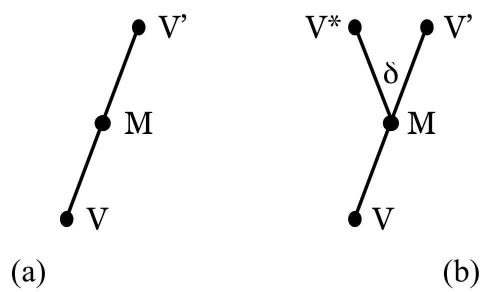

Figure 1. The basic steps in the constructions described.

we are given two points M and V, but now also a positive real number δ; we mirror V in M to obtain V', and then rotate V' about M through the angle δ resulting in V*. The angle δ (measured in radians or degrees) is called the *deflection* of the construction; as is customary, deflection is defined as the angle between the extension of an edge and the next edge. This can be formalized as

$$V^* = M + (V - M)e^{(\delta+\pi)\,i} = M + (V - M)\Delta \qquad (*)$$

where the points are taken in the complex plane and $\Delta = e^{(\delta+\pi)i}$.

Clearly, the topic of [2] corresponds to the deflection $\delta = 0$; from now on we shall assume $\delta \neq 0$.

Given a polygon $P = [M_1, M_2, \dots, M_n]$ and a deflection δ, in analogy to the procedure in [2], we start with a point $V = V_0$ and construct a sequence of points by applying the basic step to V_{j-1} and M_j to obtain V_j, $j = 1, 2, \dots$, and with subscripts of the vertices M_j reduced mod n. When appropriate, we may extend the construction backwards, to obtain a 2-way sequence of V_j's.

As illustrated in Figures 2 and 3, the sequence of V_j's appears to jump all over the plane. However, we shall see that there is an interesting order in the sequence.

Simple computations yield:

$V_1 - M_1 = (V_0 - M_1)\,\Delta$,

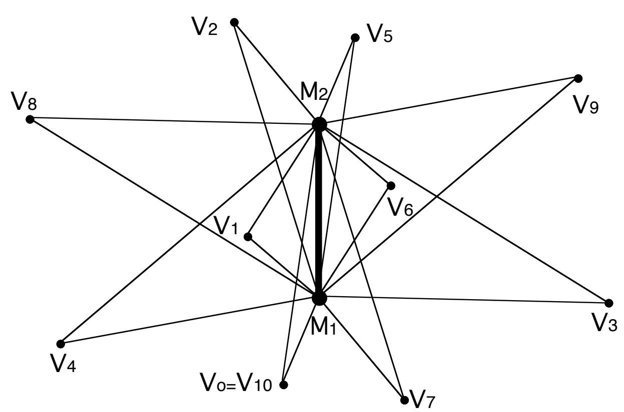

Figure 2. Illustration of case n = 2, deflection $\delta = 72° = 2\pi/5$; only some of the V_j's are labeled.

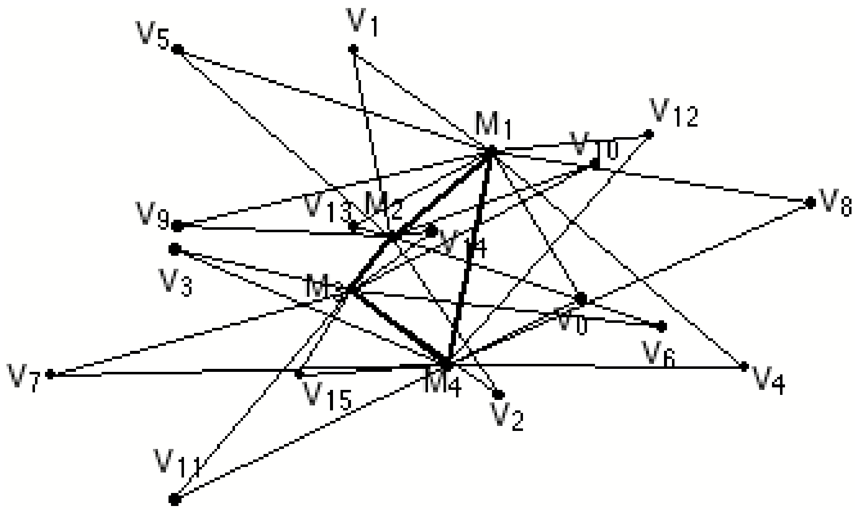

Figure 3. Illustration of case n = 4, deflection δ = 22.5° = π/8.

$V_2 - M_2 = (V_1 - M_2)\,\Delta = (V_0 - M_1)\,\Delta^2 + (M_2 - M_1)\,\Delta$

giving by induction, for all $k \ge 1$,

$V_k - M_k = (V_0 - M_1)\,\Delta^k + \sum_{1 \le j \le k-1} (M_j - M_{j+1})\,\Delta^{k-j}$.

It follows that

$$V_n - V_0 = (V_0 - M_1)\Delta^n + \sum_{1 \le j \le n-1}(M_j - M_{j+1})\Delta^{n-j-1}(\Delta - 1) + M_1\Delta - V_0, \qquad (**)$$

hence

$$V_{2n} - V_n = (V_n - M_1)\Delta^n + \sum_{n+1 \le j \le 2n-1}(M_j - M_{j+1})\Delta^{2n-j-1}(\Delta - 1) + M_1\Delta - V_n.$$

But $M_{n+j} = M_j$, thus

$$V_{2n} - V_n = (V_n - M_1)\Delta^n + \sum_{1 \le j \le n-1}(M_j - M_{j+1})\Delta^{n-j-1}(\Delta - 1) + M_1\,\Delta - V_n.$$

Therefore

$$V_{2n} - V_n = (V_n - V_0)\Delta^n\,. \qquad (**)$$

The equation (**) has a very simple meaning. We denote $W_j = V_{nj}$ and interpret the sequence W_j as the vertices of a polygonal line $Q = [W_0, W_1, \ldots, W_j, \ldots]$. Then (**) and (*) imply that all edges of Q have the same length, and that at each vertex the deflection is the same – namely, $n\delta$ or $n\delta + \pi$ depending on whether n is even or odd. Therefore, the polygonal line Q either repeats after a finite number of steps, or else it never repeats. The former happens if δ is a rational multiple of π, otherwise we have the second possibility. Hence:

(iii) If δ is an irrational multiple of π, then the sequence Q is a *cyclic apeirogon*. By this we mean a concyclic infinite sequence of points, adjacent points of the sequence being at a constant distance. This denumerable sequence is dense in the circle – hence not representable in a graphically meaningful way, – but it is of special character due to the equidistance of the adjacent pairs. Also, see the exception discussed in (e) below.

(iv) If $\delta = \pi q/r$, where q/r is a fraction in reduced form, Q is an equilateral polygon with deflection at each vertex constant and equal to $\pi nq/r$ or $\pi(1+ nq/r)$, depending on whether n is even or odd. Thus Q is a regular polygon, of a certain type {k/d}. Again, there is an exception discussed below in (e).

This is illustrated in Figures 4, 5, 6 and 7. More precisely, by equation (**), for even n the deflection at each vertex of {k/d} is $2\pi d/k$, hence we have $2d/k = nq/r$, or $k/d = 2r/nq$, so Q is the polygon {2r/nq}. For odd n we have $2d/k = 1 + nq/r$, hence $k/d = 2r/(nq + r)$ and Q is the polygon {2r/(nq + r)}.

Naturally, we may interpret the vertices of Q as either an infinite sequence of vertices that is periodic with period 2r so that each vertex of Q represents infinitely many points of the sequence of W_j's, or else consider just one period of this sequence. However, even in the latter case, there may be repeated vertices. For example, if $n = 3$ and $\delta = 24° = 2\pi/15$, the $k/d = 30/21$, and the 30 vertices of Q are represented by the 10 vertices of {10/7}, each accounting for three of the 30 vertices of Q. For more details about polygons {k/d} with k and d not coprime see, for example, [1].

Several comments seem appropriate and are illustrated by the figures.

(a) Each of the points V_i can be interpreted as leading to a polygon Q_i congruent to $Q = Q_0$. Thus the complete picture contains n polygons Q_i.

(b) In case {k/d} = {2} the polygons Q_i have to be interpreted as digons, each represented by a segment.

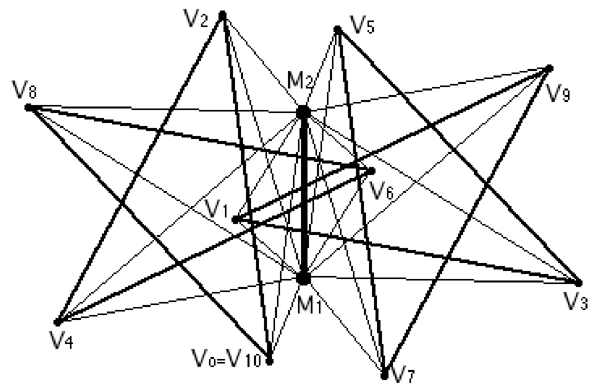

Figure 4. The rather chaotic appearing sequence of points V_j generated by the deflection construction on a digon $[M_0, M_1]$ with d = 72° = 2π/5 (shown in Figure 2) leads to a pair of regular pentagrams Q_i.

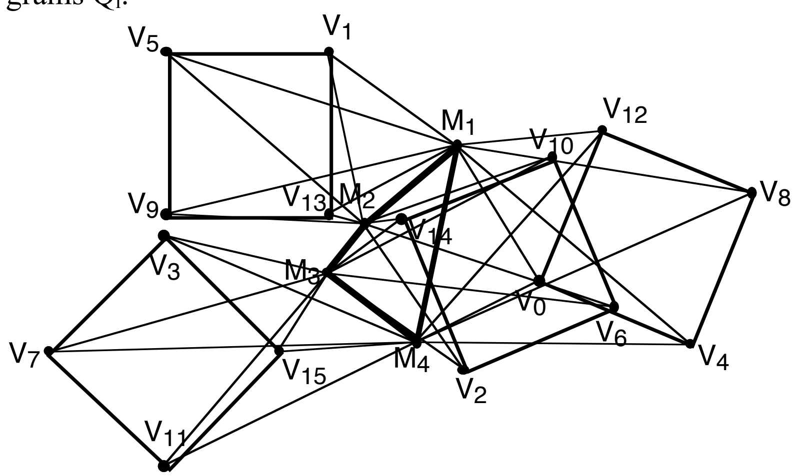

Figure 5. The sequence of the W_j's in case of n = 4 and deflection δ = 22.5° = π/8 leads to four squares Q_i.

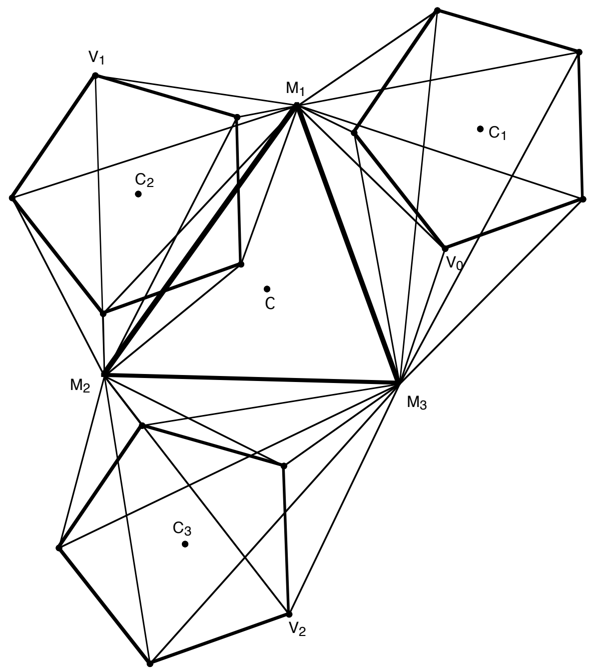

Figure 6. If n = 3 and δ = 36° = π/5, the polygons Q_i are regular pentagons.

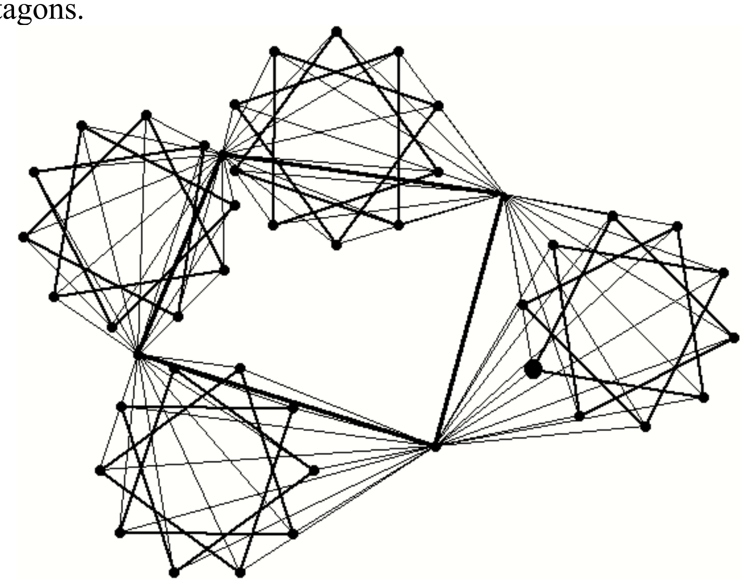

Figure 7. For n = 4 and δ = 27° = 3π/20, the resulting polygons Q_i are decagrams {10/3}.

(c) The original sequence $V_0, V_1, V_2, \dots$, can also be considered periodic, with period kn.

(d) The centers C_i of the polygons Q_i are independent of the choice of V_0, and depend only on the n-gon P. Taking C_0 as V_0, the resulting polygonal line $[V_0, V_1, V_2, \dots]$ closes after only n steps. This can be interpreted as meaning that with period kn for the vertices V_i, each of the n polygons Q_i shrank to a single point.

(e) For each n there is a singular value of δ, for which the above applies only in a modified (or limiting) way; this is illustrated in Figures 8 and 9. The singular value is $\delta = \pi/n$ for odd n, and $\delta = 2\pi/n$ for even n. What happens in the singular cases is that instead of the polygons formed by the W_j points, they are equidistant on (straight) rays – forming n what may be called *apeiro-rays* or, if extended backwards, apeirogons.

These apeiro-rays are equi-inclined, and their directions and the step (equal on all) is determined by P, while their position depends on the starting point $V_0 = W_0$. For even n the rays come in anti-parallel pairs. Also, in case P is a regular polygon, the step is of zero length, so each apeiro-ray collapses to a point (of infinite multiplicity), and the resulting n points can be interpreted asof zero length, so each apeiro-ray collapses to a point (of infinite multiplicity), and the resulting n points can be interpreted as being a sequence of period n, as illustrated for n = 4 in Figure 10.

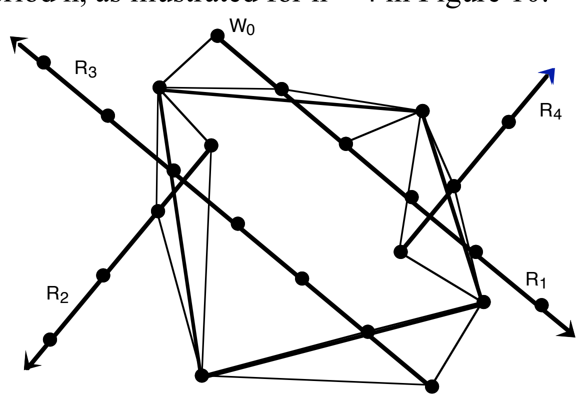

Figure 8. The case n = 4 and $\delta = \pi/2$ leads to two pairs of anti-parallel apeiro-rays.

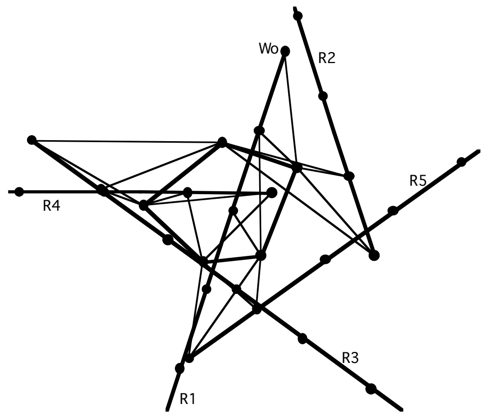

Figure 9. The five apeiro-rays in case n = 5 and δ = π/5.

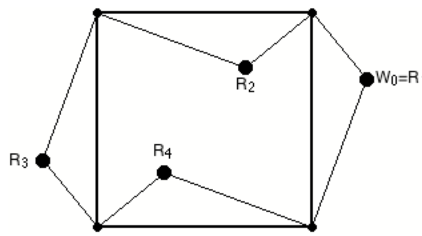

Figure 10. For regular polygons (here n = 4), in the singular case δ = 2π/n the apeiro-rays collapse to a single point each, indicated by the large dots.

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References.

[1] B. Grünbaum, Polygons: Meister was right and Poinsot was wrong but prevailed. *Beiträge zur Algebra und Geometrie* 53 No.1(2012), 57 – 71.

[2] B. Grünbaum, Inversion of the "midpoint polygon" construction. *Geombinatorics* 21(2012), 89 – 96.

[3] B. Grünbaum, Midpoint polygon inversion revisited. *Geombinatorics* .