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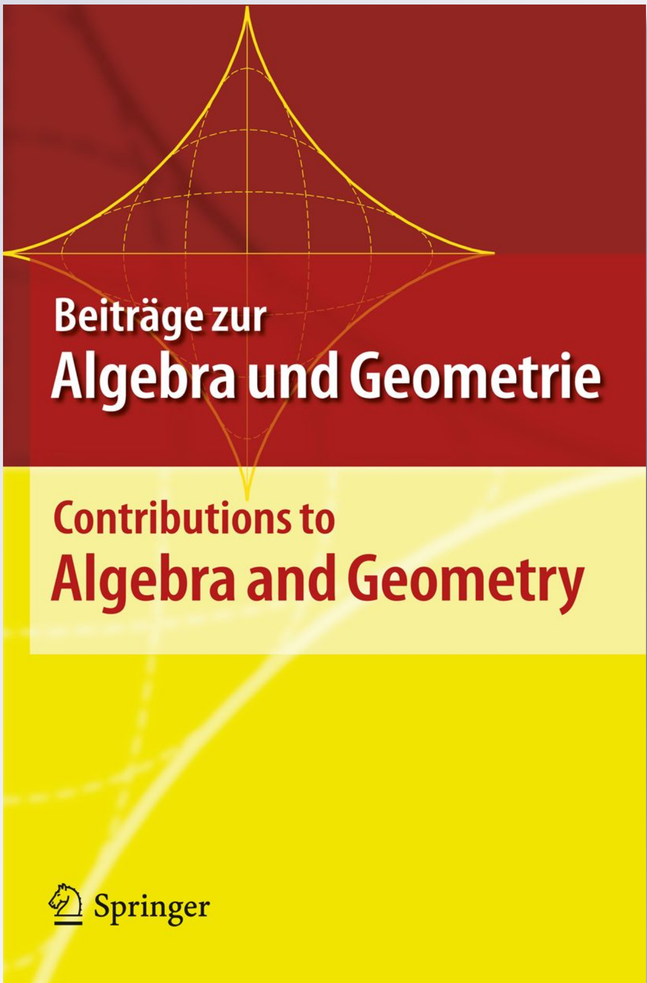

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ORIGINAL PAPER

Polygons: Meister was right and Poinsot was wrong but prevailed

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Abstract The definitions of the term "polygon" as given and used by Meister (1724–1788) in 1770 and by Poinsot (1777–1859) in 1810 are discussed. Since it is accepted that mathematicians are free to define concepts whichever way they like, the claim that one of them is right and the other wrong may appear strange. The following pages should justify the assertion of the title by pointing out some of the errors and inconsistencies in Poinsot's work, and—more importantly—show the undesirable and harmful consequences resulting from it.

Keywords Definition of polygon · Meister · Poinsot

Mathematics Subject Classification (2000) Primary 52; Secondary 01

1 Introduction

Although they use different words, both Meister (1769/1770) and Poinsot (1810) formulate their definitions essentially in the same way as most people do today, as illustrated in Fig. 1.

> A *polygon* consists of a cyclically ordered sequence of points (vertices) together with the segments determined by vertices adjacent in the cyclic sequence.

To be precise, we use labeled points, with the labels organized in a cyclic sequence. Note, however, that in the present discussion—as in the papers under consideration—only unoriented polygons appear.

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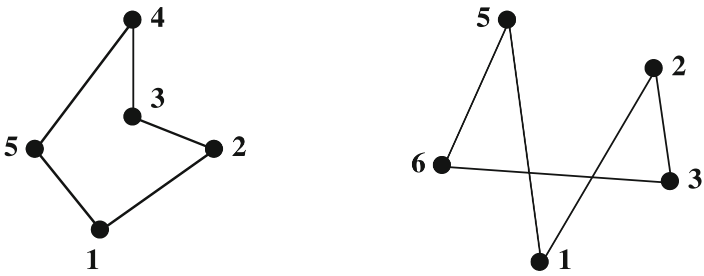

Fig. 1 Two polygons

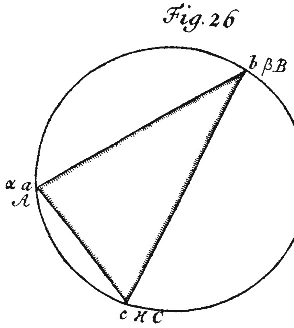

Fig. 2 An enneagon abc$\mathcal{ABC}\alpha\beta\kappa$, from Meister (1769/1770)

Meister goes to considerable lengths to stress that the vertices *need not be distinct*. He shows (see Fig. 2, taken from Meister 1769/1770, Fig. 26) what he describes as starting with three coinciding triangles, and forming a polygon with nine vertices abc$\mathcal{ABC}\alpha\beta\kappa$ by concatenating the triangles appropriately. In contrast, Poinsot and many later writers are either silent regarding the possibility of some of the vertices coinciding but exclude it by implication (Brückner 1900, p. 2; Wiener 1864, p. 1; Steinitz and Rademacher 1934, p. 20; Coxeter 1969, p. 37; Coxeter 1973, p. 94), or else explicitly restrict polygons to have distinct points as vertices (Steinitz 1922, p. 7; Cundy and Rollett 1961, p. 84; Cromwell 1997, p. 249).

Concerning *regular* polygons with n vertices, various definitions have been used by different writers. But again almost everybody agrees (Meister 1769/1770, p. 164; Poinsot 1810, p. 21; and later writers such as Cundy and Rollett 1961, pp. 83/84; Coxeter 1969, p. 36) that the various definitions are equivalent to the one represented by the following construction (as understood by Meister; to simplify the exposition, the symbols used by different authors have been replaced by a consistent notation.):

> Let a sequence of n points be given, equally distributed on a circle, with adjacent points separated by an arc spanning the central angle $2\pi/n$. For an integer d with $1 \leq d \leq n/2$, start with one of the points and select as the next the point separated from it by an angle of d times $2\pi/n$, and continue in the same way for a total of n steps. These points (in the cyclic sequence they were chosen) are the vertices of a regular polygon denoted $\{n/d\}$; the edges of this polygon are the line segments determined by vertices adjacent in the sequence.

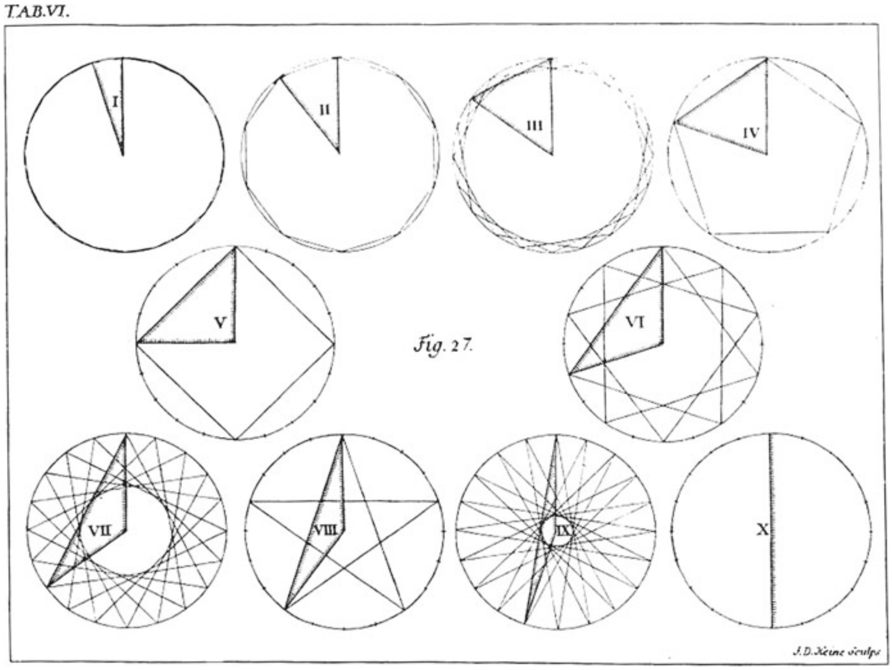

Fig. 3 Meister's illustration of the ten icosagons

To elucidate his definition Meister shows (his Fig. 27, our Fig. 3), the *ten* regular polygons $\{20/d\}$, for $d = 1, 2, \ldots, 10$; both Meister and Poinsot agree that $\{n/d\}$ is the same (unoriented) polygon as $\{n/(n-d)\}$, so $d \leq n/2$ can be assumed without loss of generality. Meister stresses that several of them *appear* as polygons with fewer sides, but that they are indeed 20-sided, with some *distinct* vertices and edges represented by the *same* points and segments. Moreover, he fully explains for what values of n and d this happens, and what is the apparent form in each case. At the end, he repeats that all these are indeed regular polygons with n vertices, $n = 20$ in the diagram. (I am indebted to the late Prof. Paul Pascal of the University of Washington for confirming my understanding of the relevant parts of Meister's text and providing a translation of them.)

In contrast, Poinsot describes his construction *only* for the case of n and d coprime (that is, without common divisors greater than 1). Hence he has only four polygons for $n = 20$. The same restriction on n and d is imposed by all the other writers mentioned, usually with the cursory explanation that otherwise one does not obtain a polygon with n vertices. Later in his paper (p. 28) Poinsot gives an explanation of sorts for the exclusion of pairs n and d with a common divisor $q > 1$. He states that connecting the n points (presumably equidistributed on a circle, separated by arcs of $2\pi/n$) by steps of length d, only n/q will be reached. Clearly, Poinsot means "distinct points". In the next paragraph Poinsot makes it clear that all n points are joined d by d. He reinforces that statement by mentioning the example (Poinsot 1810, p. 28) of joining $n = 18$ points by steps of $d = 4$ one reaches only $n/q = 18/2 = 9$ points. Obviously, this is not correct, since such joining produces *two* separate polygons, *each* with nine sides. However, in the analogous case $n = 6, d = 2$, which is explicitly discussed by Poinsot (1810, p. 26 and illustrated in the appended plate), he reaches the conclusion that there result *two* triangles, hence not a hexagon.

The important point here is Poinsot's failure to notice that this does not follow his own instructions for constructing a polygon with n sides, namely *executing n consecutive steps* (of size d). (The same inconsistency occurs in Steinitz and Rademacher 1934, p. 20). It seems clear that Poinsot shied away from having distinct vertices represented by the same point. It is also possible that he understood "... n points ..." to mean "... n distinct points ...". I do not know what the conventions and understandings in the French language were two centuries ago. Poinsot (1810) uses expressions like "Let m points ... be placed at will in a plane ..." (on p. 18) or "... points placed at will in space ..." (on p. 28) while clearly not allowing placement "at will". He does not consider the situations in which some points are collinear, or coplanar in space—but never mentions this restriction. Nor does he mention that the points should not coincide. The assumption that "m points" means "m *different* points" certainly is not a generally accepted convention. If asked to count the letters in this paragraph, nobody will understand this as a request to count *different* letters contained in it. If the latter is desired, it has to be explicitly stated.

In contrast, Dostor (1880, p. 346) states that if n and d have a greatest common divisor q, the polygon formed by joining the points in steps of d will have n/q sides and will circle d/q times. This is somewhat closer to Meister's interpretation than to Poinsot's. Unfortunately, Dostor does not return to this view in the rest of the paper, considering instead the q polygons with n/q sides that form a compound polygon.

It is strange that none of the authors mentioned so far found any reason to object to Poinsot's arguments. Even more remarkably, although seemingly accepting the requirement that distinct vertices need be represented by distinct points, some of the same authors do consider in their writings polygons in which distinct vertices are represented by a single point. For example, Hess (1876, p. 34) constructs a polyhedron all faces of which are congruent to the polygon shown in Fig. 4a, in which *five* points represent all the vertices of a *decagon*. Brückner (1906) constructs a polyhedron in which all faces are hexagons congruent to the one shown in Fig. 4b, in which one

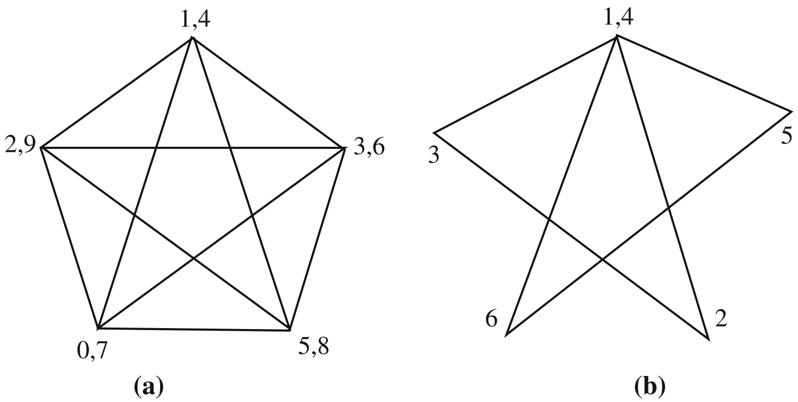

Fig. 4 Examples of polygons with distinct vertices represented by the same point, considered by Hess (1876) and Brückner (1906). The *numerals* indicate the cyclic order of the vertices

point represents two vertices. Thus, these authors have no problem allowing distinct vertices to fall on the same point—but do not even try to consider regular polygons with the same feature.

The mystery of this failure to follow Meister's lead instead of Poinsot's finds a partial explanation in the fact that most (if not all) of these authors were not aware of Meister's presentation. Although Meister's paper is mentioned quite often in very complimentary ways, it seems that few—if any—of the writers even just looked at the paper. Instead, they appear to have taken at face value the information about Meister's paper given by Günther (1876). Siegmund Günther was a prolific and respected mathematician and historian of mathematics. Unfortunately, in his account of the history of research on polygons, he misunderstood and misrepresented Meister's work, and made it appear (by misquotation, on p. 46 of Günther 1876) that Meister's attitude towards regular polygons is the same as Poinsot's. This error is particularly hard to understand, since he discusses (on p. 45 of Günther 1876) Meister's example mentioned above (an enneagon obtained by concatenating three coinciding triangles).

2 Consequences of Poinsot's definition

One may deem the inconsistencies in a paper written more than two centuries ago, and in particular the question whether n and d need or do not need to be coprime, to be a minor matter—really just splitting hairs. However, the distinction has far-reaching consequences. One is the possibility offered by Meister's approach of enlarging the family of regular polyhedra, uniform polyhedra, or other families of polyhedra. This was explored in several of my publications, such as Grünbaum (1994a,b,c, 2003a,b).

On the other hand, one of the consequences of Poinsot's approach is a breakdown of continuity in situations in which one would expect continuity. Consider the example of the continuous family of quadrangles in Fig. 5. Continuity is preserved if the rightmost apparent "triangles" are understood as quadrangles with two vertices represented by the same point. However, the Poinsot interpretation leads here and elsewhere to various difficulties and strange phenomena. For a more serious example, consider Fig. 6; the numbers under each 14-gon are values of a continuous parameter used in its

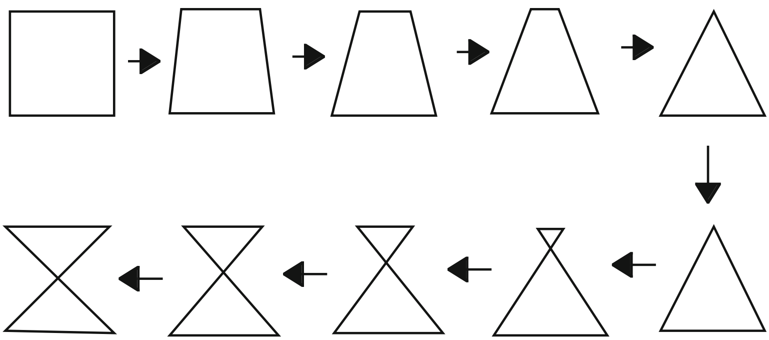

Fig. 5 A family of continuously changing quadrangles, from a square to a "crossed square"

construction, see Grünbaum (1994c). If vertices are required to be distinct, the continuous family of isogonal (equivalent under distance-preserving symmetries) hexagons is split into two parts, and that of 14-gons is split into four parts, separated by figures that in Poinsot's interpretation are either triangles or 7-gons, or not polygons at all. The adherence to the Poinsot view resulted in the inability of Hess (1874) to develop a coherent theory of isogonal polygons in a work of well over a hundred pages. In fact, the description of all such n-gons can be formulated in a few lines, since they fall into a small number (at most $n/2$) of well-defined continuous families (see Grünbaum 1994c). In an influential survey, (Steinitz 1922, p. 7) goes beyond Poinsot by not only excluding the possibility of distinct vertices falling on the same point, but also insisting that no three edges meet at relatively interior points. This would outlaw one additional hexagon and two additional 14-gons in Fig. 6, and would split the families into three or six parts!

A different drawback of the Poinsot approach is the difficulty of formulating various theorems about polygons in a general way. As a typical example we may mention the theorem generally associated with van Aubel (1878) (although it was published earlier by Laisant 1877; this is another example of the frequently observed fact that a result is not commonly known by the name of its first discoverer.): *The centers of squares erected in consistent orientation on the sides of an arbitrary quadrangle are at vertices of a quadrangle with equal diagonals, perpendicular to each other.* This result remains valid even if some or all vertices of the starting quadrangle are collinear, or coincide. With Poinsot's interpretation there would be a large set of distinct theorems, some of which are illustrated in Fig. 7.

Coxeter et al. (1953) enumerate uniform polyhedra by certain constructions (including truncation), see Figs. 8, 9, 10. However, in two cases (among the regular Kepler–Poinsot star polyhedra) they dismiss the fully truncated forms since "the truncation of {5/2, 5} consists of three coincident dodecahedra, while the truncation of {5/2, 3} consists of two coincident great dodecahedra along with the icosahedron that has the same vertices and edges". This assertion goes back to Coxeter's (1931) paper (reprinted in Sherk et al. 1995, p. 43) and the claim that the truncations of a pentagram end in two coinciding pentagons. But after I sent Coxeter the diagrams in Figs. 8 and 10, with the explanation that the final result of truncating the pentagram is a decagon {10/2}, he answered by email on 5/26/2001, "... Your drawings of the process of truncation are beautiful and convincing". The final truncation of {5/2, 5} is the uniform polyhedron (5. 10/2. 10/2), that of {5/2, 3} is (3. 10/2. 10/2).

Another example concerns isohedra with equilateral triangles as faces, studied by Shephard (1999). One of the constructions he employs is the replacement of each face of a regular polyhedron by the mantle (consisting of equilateral triangles) of a pyramid. Shephard states that this construction can be carried out on eight of the nine regular (Platonic and Kepler–Poinsot) polyhedra, but that it is not applicable to {5, 5/2}. This is not correct, since by continuity the resulting polyhedron is an isohedral hexecontahedron of type [6. 10/2. 10/2], *isomeghetic* (that is, of equal extent) with the icosahedron, and each icosahedral triangle is covered by three combinatorially distinct triangles of the hexecontahedron. Shephard also claims that the analogous operation of "excavating" pyramids on faces of the nine regular polyhedra fails in

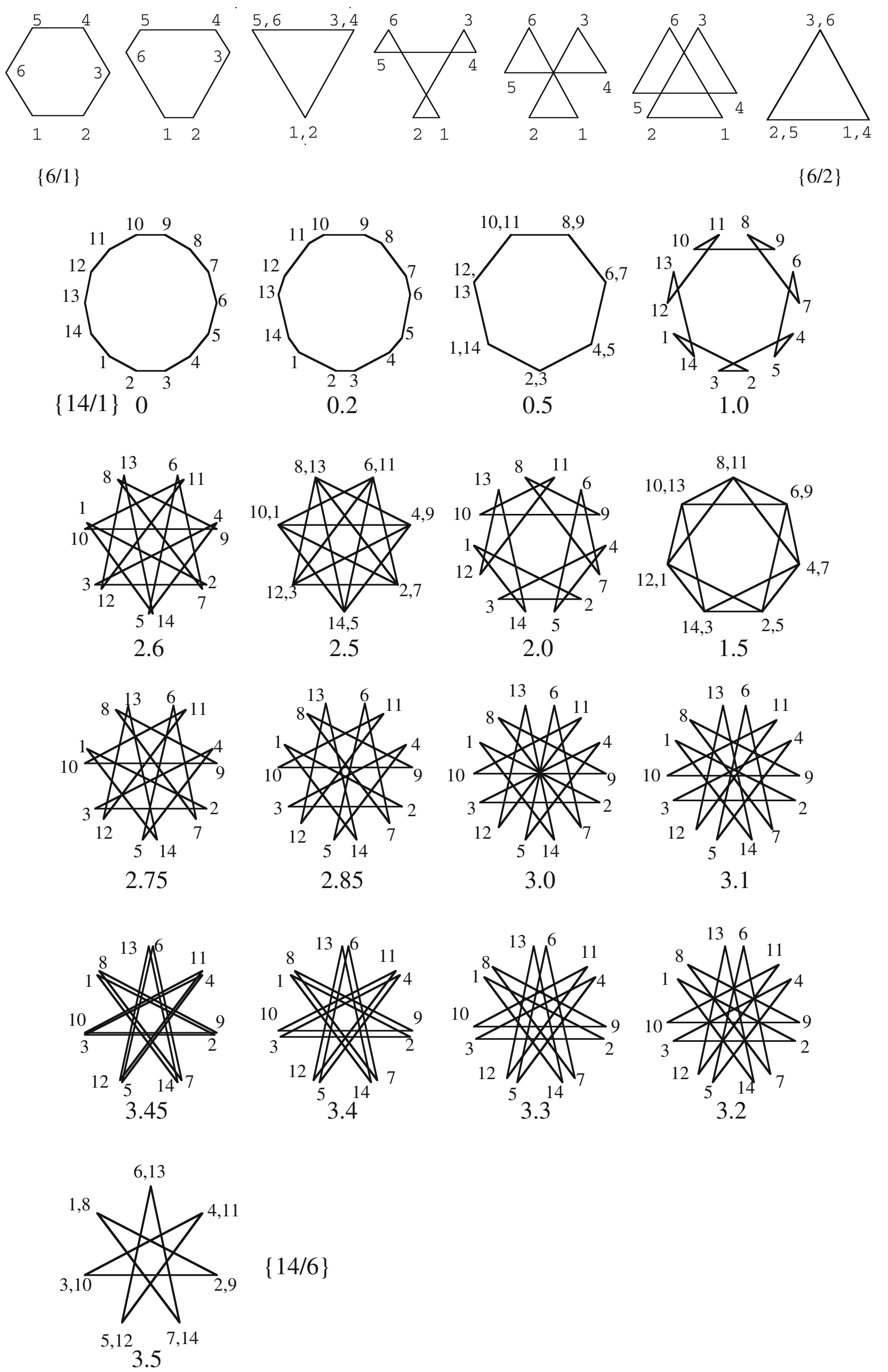

Fig. 6 Continuous families of hexagons and of 14-gons, that would be split into two (resp. four) parts if coinciding vertices were not admitted, and into three (resp. six) parts if Steinitz' restrictions are accepted

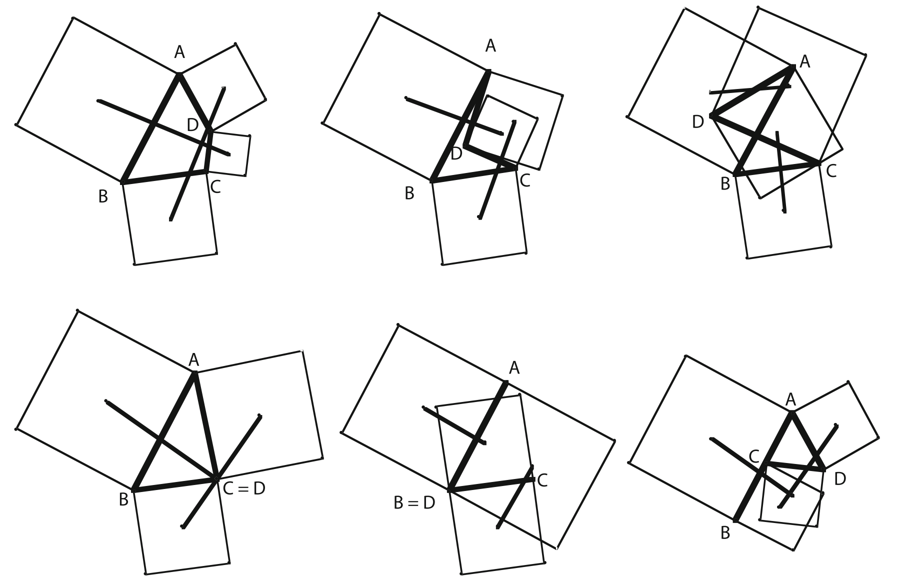

Fig. 7 Illustrations of few of the varied possibilities of the so-called van Aubel theorem. The starting quadrangle is shown by *heavy segments* and labeled ABCD; the two segments determined by centers of squares erected on opposite sides of the quadrangle are shown by less *thick segments*; in all cases these are of same length, and perpendicular to each other

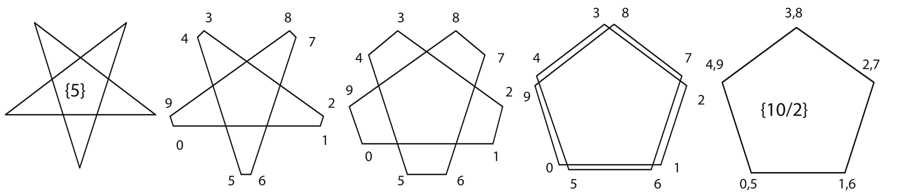

Fig. 8 Truncations of the pentagram, leading to the decagon {10, 2}

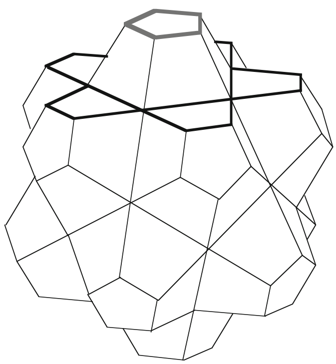

Fig. 9 A truncation of the Kepler–Poinsot regular star-polyhedron {5/2, 5}

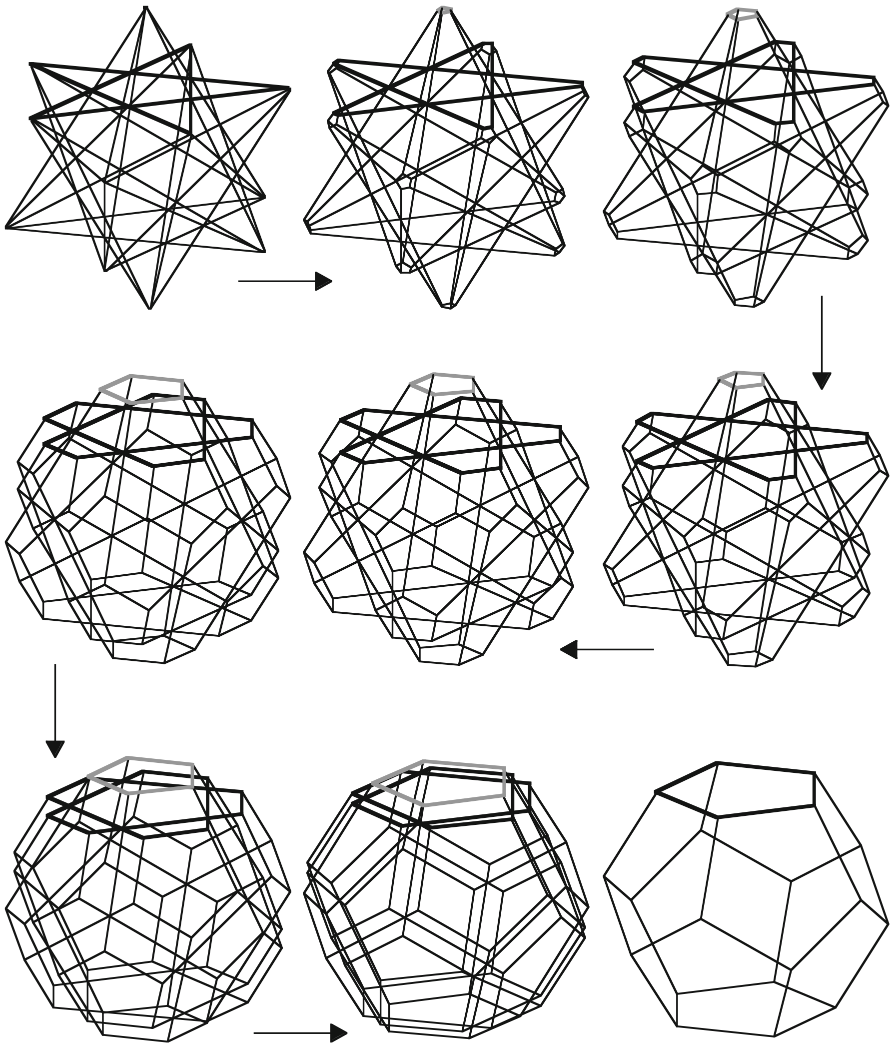

Fig. 10 The final truncation of {5/2, 5} is the uniform polyhedron {5. 10/2, 10/2}

one case, the case of the tetrahedron; the failure being explained by the misleading statement that "... since the construction leads to a set of twelve equilateral triangles which coincide in sets of three." This disregards the combinatorial structure of the resulting isohedral polyhedron [3. 6/2, 6/2] that is, in fact, combinatorially equivalent to the isohedral polyhedron [3. 6. 6] obtained by Shephard's first method (of erecting pyramids).

Rejection of Poinsot's approach to polygons and, in particular, to regular polygons, and independent rediscovery of Meister's approach, was an indispensable prerequisite for the body of work that has come to be known as "relatives and extensions of Napoleon's theorem". The attribution of any theorem of this kind to Napoleon

Bonaparte is undocumented and quite dubious. The mathematicians involved did not dwell on the history (neither Meister nor Poinsot is mentioned in any of these papers), but just used Meister's way of looking at polygons. Early papers in this vein are those by Douglas (1940), Neumann (1941), Schoenberg (1950), and Berlekamp et al. (1965), while some more recent ones are Fisher et al. (1981), Neumann (1982), Martini (1996), Schuster (1998), and Shephard (2003).

To conclude, here is the only reference to the difficulties in Poinsot's approach that I was able to find in published works (other than my own). It appears in the entry "Poinsot" by O'Connor and Robertson (2010).

> He wrote an important work on polyhedra in 1809 (already mentioned above), discovering four new regular polyhedra, two of which appear in Kepler's work of 1619 but Poinsot was unaware of this. In 1810 Cauchy proved that, with this definition of regular, the enumeration of regular polyhedra is complete. A mistake was discovered in Poinsot's (and hence Cauchy's) definition in 1990 when an internal inconsistency became apparent.

This is strange for at least two reasons. First, there is no "mistake" in the definition; it is equivalent to the one given above. The problem is that Poinsot does not adhere to his own definition in deciding what regular polygons exist. Second, how can a reader find out anything about the alleged "mistake", when no explanation or reference is given.

3 Comments: on terminology and other aspects

(i) Poinsot's attitude throws a long shadow. For example, in many publications it is stressed that various formulas are valid for regular polygons $\{n/d\}$ for relatively prime n and d. One of these is the mention by Coxeter (1969, p. 37) that $2R = s/\sin(\pi d/n)$, where R is the circumradius and s the length of the side of a regular polygon $\{n/d\}$ whenever n and d are relatively prime. But in this case, and in all the other cases I could think of, the relationship remains valid for *all* n and d.

(ii) As mentioned earlier, Poinsot described an attempt to construct the regular polygons $\{n/d\}$ in case n and d have a greatest common factor $f > 1$. The method consists of placing n equidistributed points on a circle, and connecting by a segment each point with the point d steps away on the circle. This is a perfectly reasonable construction—however, it does not yield a polygon, regular or not. It gives f polygons of type $\{n^*/d^*\}$, where $n^* = n/f$ and $d^* = d/f$. This was recognized by Poinsot with respect to $\{6/2\}$, but not applied clearly in case $\{18/2\}$. Such objects are best understood as *compounds of polygons*, in analogy to the better known compounds of polyhedra (such as Kepler's *Stella Octangula*, or five tetrahedra in a dodecahedron; see Cundy and Rollett 1961, pp. 129, 139; Cromwell 1997, Plates 11, 12). A convenient notation (for the regular compounds) would be $f\{n^*/d^*\}$ using the above symbols. The same construction is described in various other places; for example, in the MathWorld "Star Polygon" entry (Weisstein 2010a). Unfortunately, the symbol $\{n/d\}$ is said in the same entry to be "*a star polygon-like figure*", also called "*a star figure*". (However, in Heckman and Weisstein 2010 we read that "The hexagram is the star polygon {6/2}...") The star figure terminology seems to have been adapted from Savio and Suryanarayan (1993), where the "regular star-figure" is taken to be the general concept, which specializes to star-polygons for n and d coprime. Not surprisingly, Savio and Suryanarayan (1993) find that some relations involving Chebychev polynomials are valid for all "regular star-figures"—but neither they nor Weisstein notice that the same relations are valid for *all* regular polygons $\{n/d\}$ and *therefore* hold for the compound polygons. The "Star polygons" article in Wikipedia (2010) presents a very unclear picture as to what are star polygons; various mutually incompatible definitions are mentioned in the same paragraph. To quote:

> "[A] **regular star polygon** is a self-intersecting, equilateral and equiangular polygon, created by connecting one vertex of a simple, regular, p-sided polygon to another, non-adjacent vertex and continuing the process until the original vertex is reached again. Alternatively for integers p and q, it can be considered as being constructed by connecting every qth point out of p points regularly spaced in a circular placement."

(iii) The construction of the regular compounds presented in (ii) is also described in the "Polygram" entry (Weisstein 2010b) as a generalization of regular polygons. To judge by the illustrations, and by the short table, the term excludes convex polygons {n}, although this is not stated. Another curious aspect of this entry is the statement that Lachlan (1893) "defines polygram to be a figure consisting of n straight lines". This is correct, but highly misleading. Lachlan's "polygrams" have no connection to Weisstein's "polygrams". For Lachlan a polygram is the figure formed by *any* family of *unbounded* straight lines, while Weisstein's polygrams are star polygons or star figures formed by *segments* in a very *regular* way.

(iv) Poinsot's (1810) paper is very poorly written, leading to inconsistent treatment of several topics. One of these concerns the regular compounds $2\{3\}$ and $2\{9\}$ described in (ii) above. More important is his confused treatment of the different "kinds" ("espéce") of n-gons. In Section 6 of Poinsot (1810) we read that only convex and regular polygons will be considered, with the explanation that "regular" means that all the angles are equal and all the sides are equal. But right after that Poinsot explains that even for "irregular" polygons the "kind" is determined by the sum of the angles, and mentions as example that all pentagons of one kind, the "ordinary" ones, have angle sum equal to 3π, while for the other "kind", in which the perimeter makes two turns of the angular space, the sum is just π, as for triangles. Moreover, in Section 7 Poinsot states that his claim that there are for each n precisely as many different kinds as there are numbers between 1 and $(n-1)/2$ that are relatively prime to n does not require that the polygons are *regular*, but remains valid for all *convex* polygons. But this clearly disregards polygons such as the ones in Fig. 11; each is convex in Poinsot's sense and has edges of same length. Such polygons can be made to

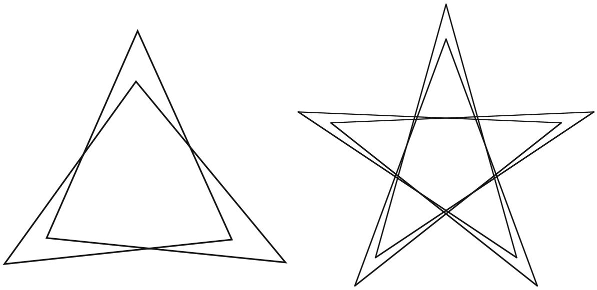

Fig. 11 A hexagon and a decagon that are convex in Poinsot's sense but have angle sum 2π. Analogous $(4k+2)$-gons can be constructed for every $k \geq 1$. For better intelligibility, they can be considered as variants of the regular $\{(4k+2)/2k\}$ polygons, arising from the "concatenation" of two coinciding $\{(2n+1)/n\}$-gons, as in Meister's example of three triangles (see Fig. 2)

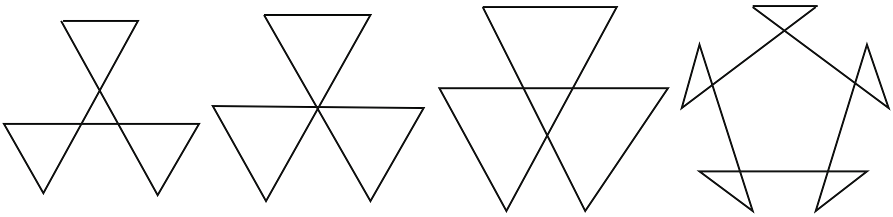

Fig. 12 Several examples of another infinite family of convex (in the sense of Poinsot) n-gons with $n \equiv 2$ (mod 4) and with angle sum 2π

approach {6/2} resp. {10/4}, but have angle sum 2π; hence their "kind" is not included in Poinsot's count. Analogous polygons $\{(4k + 2)/2k\}$ exist for all $k \geq 1$.

(v) At the end of a lengthy discussion, Poinsot concludes in Section 13 that for all *convex* n-gons, the "kinds" for which the angle sums are the smallest have angle sums equal to those of the triangle (π), the quadrangle (2π) and the hexagon (4π), respectively; moreover, Poinsot states that the triangle, the quadrangle and the hexagon are the only n-gons that are of unique "kinds". These claims stood unchallenged for two centuries, but they are trivially wrong—see Figs. 11 and 12 for a few examples with angle sum 2π. Additional examples are shown in Fig. 6, where all the 14-gons shown, except the first three, have angle sum 2π.

(vi) Naturally, the assertions in (iv) and (v) are valid if the discussion is limited to regular polygons. But then it is unclear why Poinsot stresses in each case that the polygons considered are convex—without mentioning that they are regular, which would imply their convexity. The statement on page 21 that all polygons considered are convex is discredited not only by the immediately following sentence, but also by all the material of Sections 18–25 (pp. 28–34).

4 What now?

There is no gainsaying that Poinsot's (1810) work exerted a powerful influence on the development of geometry in the two centuries following its publication, in contrast to Meister's (1769/1770) paper. Will this situation continue?

Not everything Poinsot introduced has been generally accepted. The most significant of the rejected ideas was his definition of "convex polygon". Departing from earlier understandings Poinsot wished to call "convex" any polygon for which the deflections at all vertices have the same sign. This seems to have been done in order to make his regular polygons "convex", hence more acceptable. However, this concept of convexity has been completely abolished in more recent publications. This is fortunate, in view of the tremendous development of the theory of convex polyhedra and polytopes relying on the traditional definition. The "consistent-trending" polygons that Poinsot envisaged are interesting in their own right, and deserve a more detailed study—without calling them "convex".

The difference between the approaches of Meister and Poinsot extends to polyhedra in an obvious way. Adopting Meister's approach leads to many new polyhedra, as illustrated in Grünbaum (1994b, 2003a,b), as well as to many new questions. However, as we have seen in Sect. 3, adopting Poinsot's approach is cumbersome in many situations. Even more significant is the fact that it collides with some frequently accepted ideas, such as that to every polyhedron should correspond a reciprocal one with respect to any sphere, provided its center is not on the plane of a face of the polyhedron.

A case in point is presented by the polars of the uniform polyhedra studied and enumerated by Coxeter et al. (1953/1954). According to Grünbaum (2003b), there are two distinct reasons for the failure of this idea as applied to the uniform polyhedra, to obtain isohedral ones by polarity with respect to the circumsphere:

In several uniform polyhedra some of the faces pass through the centroid of the polyhedron; therefore there is no polar *isohedral* polyhedron. Brückner (1900, p. 191) ignores the question of polars of such polyhedra, although he claims to be systematically discussing the isogonal polyhedra *and* their polar isohedral ones. Wenninger (1983) and Har'El (1993) solve the problem of polars of some of the uniform polyhedra by admitting unbounded faces. Such an approach is interesting, but it does not fall within the usual scope of the meaning of "isohedral polyhedron" or even "polyhedron". The polars exist in all these cases, but just not with respect to the circumsphere—the resulting polar polyhedra are not isohedral.

Another difficulty is that some of the uniform polyhedra have pairs of coplanar faces; hence the polar polyhedra must have pairs of coinciding vertices—which would make them unacceptable under the traditional, Poinsot derived, definition of polyhedra. It should be mentioned that neither in Wenninger (1983) nor in Har'El (1993) (nor in the reviews of these publications) is any mention made of this fact. The vertices which are incident with two cycles of faces are neither noticed nor explained, nor is any mention made of the fact that, for example, the uniform polyhedron (3.3.3.3.3.5/2) has 112 faces, but the purported polar shown in Wenninger (1983) and Har'El (1993) has only 92 vertices. On the other hand, in the interpretations of polyhedra in the sense of Meister there is no problem in such cases: the two vertices of each pair are distinct, and only in the realization they happen to be represented by a single point. In contrast to the preceding, the difficulty of the traditional approach cannot be eliminated by changing the sphere that yields the polarity.

It seems quite clear that any serious discussion of polygons and polyhedra, regular or not, needs to pay attention to Meister's approach. Failure to do so leads to complications in some cases, and to inconsistencies (or errors) in other cases. While one has to admit that Poinsot's approach makes for simpler exposition, the treatment of the Meister interpretation presented in Grünbaum (2003a,b) is not very complicated. The extent to which this, or some other Meister-inspired definition of polyhedra is adopted depends on the scope of the publication. In restricted contexts it may be sufficient to rely on Poinsot's concepts, provided it is noted that they encounter difficulties that may be alleviated by the more general approach of Meister's.

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