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DELETION CONSTRUCTIONS OF SYMMETRIC 4-CONFIGURATIONS. PART I.

LEAH WRENN BERMAN AND BRANKO GRÜNBAUM

Abstract. By *deletion constructions* we mean several methods of generation of new geometric configurations by the judicious deletion of certain points and lines, and introduction of other lines or points. A number of such procedures have recently been developed in a systematic way. We present here one family of such constructions, and will describe other families in the following parts.

1. Introduction

Geometric 3-configurations—that is, families of points and lines in the Euclidean or real projective plane, such that each point is on three lines and each line passes through three of the points—have been studied for a century and a quarter, with a variety of results known about them. In contrast, analogously defined 4-configurations have a much shorter history, and the first visually intelligible example was published less than twenty years ago [10]. (If the number of points (and lines) n of a 4-configuration is relevant for a discussion, the configuration is referred to as an (n_4) configuration.) In the short period since [10] was published, a large number of methods of construction of such configurations have been found, and the advances have been in many directions (see, e.g., [1, 2, 3, 4, 5, 7]). A recent survey appears in [8], and a detailed account of many of the results in the theory of all geometric configurations has just been published [9].

Many of these configurations in the Euclidean plane exhibit a very high degree of symmetry, in the sense that isometric maps of the plane map the points and lines of the configuration on themselves. This is easily seen to imply that the number of points (and lines) n must be a composite number, and that the symmetry group consists of rotations about a fixed point, the *center* of the configuration, and possibly reflections in mirrors through the same point. Typically, we assume that the fixed point is at the origin. In most of the constructions the number of points in each orbit (of the symmetry group) is the same, and equals the order of the symmetry group, and these points all lie on a circle whose center is the center of the configuration. We will refer to configurations with h symmetry classes of points

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as h-ring configurations. Typical examples are shown in Figures 2 and 7; both configurations in Figure 2 shown are 3-ring configurations, while the configurations in Figure 7 are 4-ring configurations.

The present paper is the first in a series devoted to a systematic development of methods of construction that yield configurations in which not all orbits need to have the same number of points (or lines). The characteristic aspects of these constructions is that they start with given geometric configurations (usually quite symmetric) and through judicious deletion of points (and/or lines) and addition of suitable elements lead to new types of geometric configurations.

2. Modified celestial configurations: the deletion technique

Celestial (n_4) configurations, named in [2], were originally developed by Branko Grünbaum [6] and further studied by Marko Boben and Tomaž Pisanski [4] and Grünbaum [8]. They are highly symmetric configurations and have the property that every point has precisely two lines from each of two symmetry classes passing through them, and if there are m points in a symmetry class, then the configuration has the symmetries of a regular m-gon; that is, symmetry group d_m.

Many (n_4) configurations may be constructed by deleting parts of certain symmetry classes of celestial configurations and replacing them with other objects. The movable (n_4) configurations discussed in [2] were constructed in this way, yielding configurations with only rotational symmetry. Several other classes of modified celestial configurations will be discussed here. To present these constructions we need notation for the celestial configurations involved. We describe these first.

The most recent notation for celestial configurations is discussed thoroughly in [2], based on the treatment in [8]; an outline is as follows.

A connected celestial configuration has symbol

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h);$$

to emphasize the number of symmetry classes of points and lines, we will say that such a configuration is an h-celestial configuration (with h symmetry classes of points and of lines). It begins with a set of m points collectively called v_0 forming a regular m-gon which are labelled[1] cyclically in the counterclockwise direction as

$$(v_0)_0, (v_0)_1, \ldots, (v_0)_{m-1};$$

by convention these are centered at the origin with

$$(v_0)_i = \left(\cos\left(\frac{2\pi i}{m}\right),\ \sin\left(\frac{2\pi i}{m}\right)\right).$$

[1]The notation in [2] has been modified slightly in the present treatment; there points were labelled, e.g., as $v_{0,0}$, $v_{0,1}$, where here they are labelled as $(v_0)_0$, $(v_0)_1$, and similarly for lines.

Draw all lines of span s_1—that is, lines connecting the vertex $(v_0)_i$ with $(v_0)_{i+s_1}$—and label these consecutively as $(L_0)_1, (L_0)_1, \ldots, (L_0)_{m-1}$, or collectively as L_0. On the t_1-st intersections of these lines, which are each given the label $[[s_1, t_1]]$ (see Figure 1), counting from the center of the line segment $(v_0)_i(v_0)_{i+s_1}$, place a new set of vertices $(v_1)_0, \ldots, (v_1)_{m-1}$. Draw in lines of span s_2 (collectively labelled L_1) using these vertices, and place the third set of vertices, with label $[[s_2, t_2]]$, at the t_2-nd intersection of these new lines. Continue in this fashion until all the s_i and t_i have been used up; if the symbol corresponds to a valid configuration, the points labelled $v_h = [[s_h, t_h]]$ that are the t_h-th intersection points on the span s_h lines (which have label L_{h-1}) will coincide—as sets—with the original ring of points labelled $(v_0)_0, \ldots (v_0)_{m-1}$, although the point $(v_h)_i$ may not be the same point as $(v_0)_i$. Notice that by construction, for each i, the set of points with label v_i forms a regular convex m-gon centered at the origin.

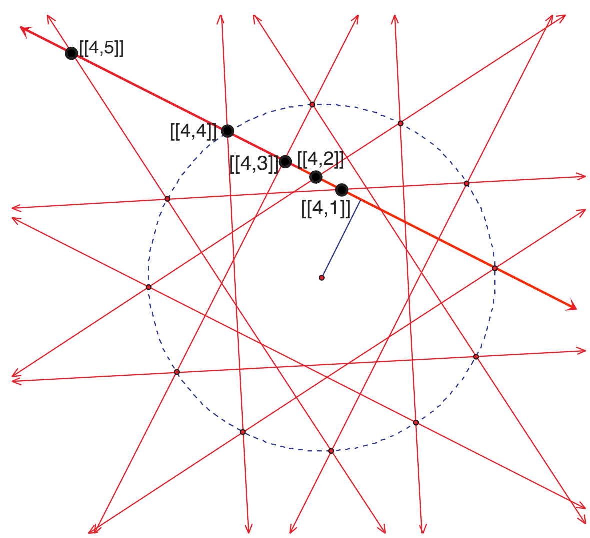
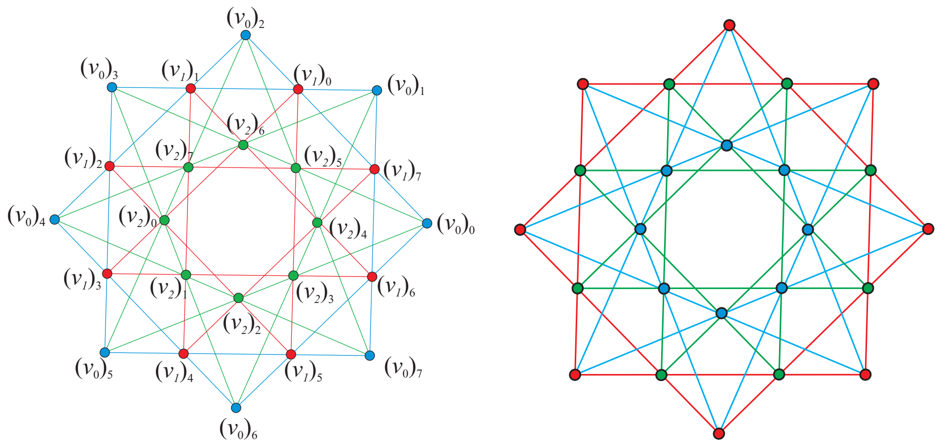


FIGURE 1. The notation $[[s, t]]$ for points on lines of span s. Here, $m = 12$, $s = 4$ and $t = 1, 2, 3, 4, 5$.

Several configuration symbols may correspond to the same geometric configuration, and the points labelled v_0 need not be the outermost ring of points; Figure 2 shows such a situation for $m = 8$. For a configuration symbol to be valid, two consecutive terms must be distinct, and there are various other constraints on the s_i and t_i as well; see [8, p. 202] for details. Of particular utility are the *trivial* celestial configurations, where $\{s_1, \ldots, s_h\} = \{t_1, \ldots, t_h\}$ as sets. For example, Figure 2 shows two trivial 3-celestial configurations, where in both cases $\{s_1, s_2, s_3\} = \{t_1, t_2, t_3\} = \{1, 2, 3\}$ as sets.

We say that a line is a *diameter* of a celestial configuration if it passes through the center of symmetry of the configuration (conventionally taken to be the origin) and a point from the set v_0. If m is even, all diameters connect



(a) $8\#(2,1;3,2;1,3)$, with points labelled

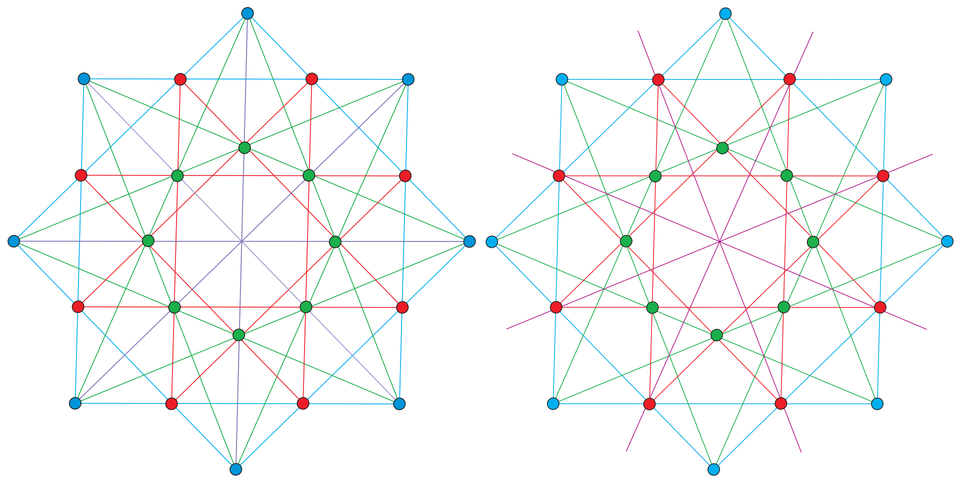
(b) $8\#(1,3;2,1;3,2)$

FIGURE 2. Illustrating notation used for celestial configurations. This shows two symbols corresponding to the same celestial configuration. In each configuration, lines L_0 and points v_0 are blue, L_1 and v_1 are red, and L_2 and v_2 are green.

pairs of points $(v_0)_i$ and $(v_0)_{i+\frac{m}{2}}$. We say a line is a *mid-diameter* if it is the rotation by an angle of $\frac{\pi}{m}$ of some diameter. (If m is odd, mid-diameters are themselves diameters.) All diameters and mid-diameters are mirrors of the configuration. If diameters (can) pass through a class of points, that class is said to be *diametral* or of *type* D, and likewise if mid-diameters (can) pass through the points they are said to be *mid-diametral* or of *type* MD. If there are two classes of points and they both are diametral or both are mid-diametral, the classes of points are the same *type*. A configuration symbol $m\#(s_1,t_1;s_2,t_2;\ldots;s_h,t_h);D$ or $m\#(s_1,t_1;s_2,t_2;\ldots;s_h,t_h);MD$ denotes a celestial configuration with diameters or mid-diameters added, respectively. Examples of diameters and mid-diameters are shown in Figure 3; in the underlying configurations, the points labelled v_0 (blue) and v_2 (green) are type D, while points labelled v_1 (red) are type MD.

Suppose that $m\#(s_1,t_1;s_2,t_2;\ldots;s_h,t_h)$ is a celestial configuration where m is even. To construct modified celestial configurations, we will need the following results from [2], somewhat restated from that context.

We wish to analyze what happens to a configuration when half of a symmetry class of lines or points (that is, every other point or line) is removed. Note that deleting half the points or lines in a symmetry class makes no sense if m is odd! For the remainder of the section, we will assume m is even, and when we say we are *removing half of the elements* of a symmetry



(a) Diameters: $8\#(2,1;3,2;1,3); D$.

(b) Mid-diameters: $8\#(2,1;3,2;1,3); MD$.

FIGURE 3. Diameters and mid-diameters of the configuration $8\#(2,1;3,2;1,3)$, shown in Figure 2(a). Neither of these incidence structures is a configuration.

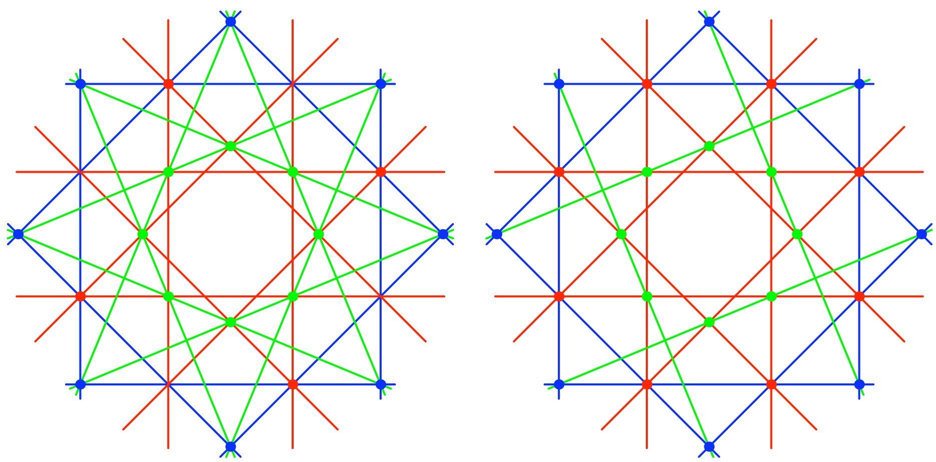
class, we mean that we are removing the elements of the class which are all of even index or all of odd index. For example, if we remove half the lines L_1 of even index, we are removing the lines $(L_1)_0, (L_1)_2, (L_1)_4, \ldots, (L_1)_{m-2}$, but if we remove half the lines labelled L_1 of odd index, we remove the lines $(L_1)_1, (L_1)_3, (L_1)_5, \ldots, (L_1)_{m-1}$.

A single symmetry class of lines L_{i-1} contains points v_{i-1} and v_i; with respect to the points v_{i-1} they are of span s_i, while with respect to the points v_i they are of span t_i. If s_i is odd, then removing half of the lines labelled L_{i-1} results in each point of v_{i-1} with still exactly one line labelled L_{i-1} passing through it. On the other hand, if s_i is even, then removing half of the lines labelled L_{i-1} means that every other point labelled v_{i-1} has both lines labelled L_{i-1} removed, while the remaining points have no lines removed.

Still considering the lines L_{i-1} but looking at them from the point of view of the points v_i so that they are of span t_i, if t_i is odd then removing half the lines L_{i-1} results in exactly one line from the symmetry class passing through each point v_i, while if t_i is even, every other point labelled v_i has no lines labelled L_{i-1} passing through it and remaining points still have two lines passing through them.

Similarly, looking backwards, if t_i is odd, removing half of the points (every other point) labelled $v_i = [[s_i, t_i]]$ means that each line L_{i-1} contains a single point labelled v_i, while if t_i is even, every other line L_{i-1} contains two points labelled v_i and the remaining lines contain no points. If s_{i+1} is odd, then removing half the points labelled v_i results in every line labelled L_i having a single point labelled v_i lying on it, whereas if s_{i+1} is even, removing half the points v_i results in every other line L_i containing two points v_i and the rest containing none.

We place an asterisk in front of t_i to indicate that half the points $[[s_i, t_i]]$ have been removed and say that the configuration is *point-deleted*; an asterisk in front of s_i indicates that half the lines of the corresponding symmetry class L_{i-1} have been removed and we say the configuration is *line-deleted*. (Note that in [2] such modified configurations were referred to as *point-modified* and *line-modified*, respectively.) Figure 4 shows examples of such modifications. Note that a point-deleted or line-deleted configuration is not typically itself a configuration, but rather merely an incidence structure.



(a) A point-deleted configuration: $8\#(2, *1; 3, 2; 1, 3)$

(b) A line-deleted configuration: $8\#(2, 1; 3, 2; *1, 3)$

FIGURE 4. Point- and line-deleted configurations. In both cases, points v_0 and lines L_0 are blue.

To modify celestial configurations so that we can add diameters or mid-diameters, we need the following observation:

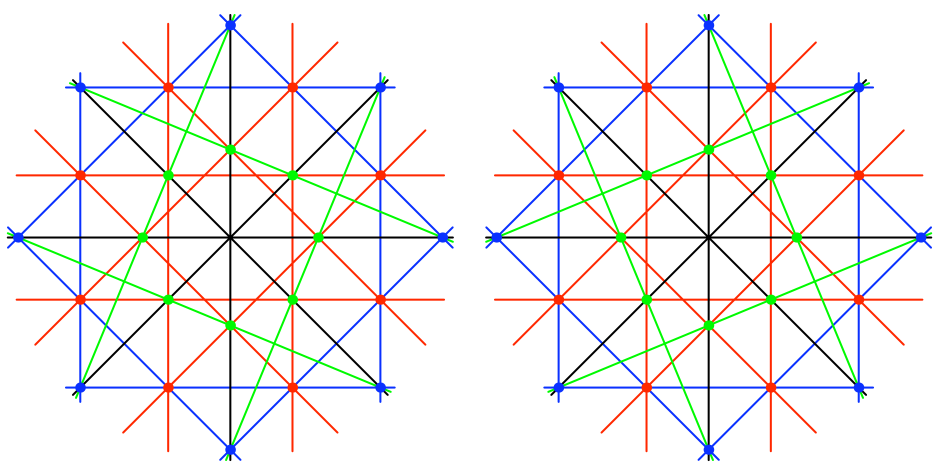
Lemma 2.1. *For a given i, if $s_i \equiv t_i \bmod 2$, the points labelled v_i (i.e., $[[s_i, t_i]]$) are the same type as the points labelled v_{i-1} (or $[[s_{i-1}, t_{i-1}]]$), with indices taken modulo h; if $s_i \not\equiv t_i \bmod 2$, then the points v_i and v_{i-1} are of opposite type.*

That is, for even m, the preceding lemma says that parity changes between s_i and t_i correspond to switching between points lying on diameters and on mid-diameters, or vice versa. For example, in Figure 2(a), the points in classes v_0 and v_1 are of opposite type, as are those of v_1 and v_2, but the points of v_2 and v_0 are of the same type. By convention, points labelled $v_0 = [[s_h, t_h]]$ are diametral.

3. Deleting lines and adding diameters: $(hm_4) \to (hm_4)$

If s_i and t_i are both odd, then two facts are true: (1) removing half the lines labelled L_{i-1} results in each point labelled v_{i-1} and v_i containing exactly one line labelled L_{i-1} (and two other lines of a different symmetry class, for a total of three lines remaining per point) and (2) the points labelled $v_i = [[s_i, t_i]]$ and $v_{i-1} = [[s_{i-1}, t_{i-1}]]$ are of the same type. Therefore, we can add in either diameters or mid-diameters, depending on the type of $[[s_i, t_i]]$, and these lines will pass through the points labelled $v_i = [s_i, t_i]]$ and $v_{i-1} = [[s_{i-1}, v_{i-1}]]$. If these are the only two points of that type, then the result will be a (n_4) configuration.

In fact, this construction technique yields two enantiomorphic configurations, depending on whether the half of the lines that were removed were of even or odd index. Figure 5 shows two examples. Both examples have symbol $8\#(2,1;3,2;*1,3);D$.



(a) Lines $(L_2)_1$, $(L_2)_3$, $(L_2)_5$, $(L_2)_7$ of odd index removed.

(b) Lines $(L_2)_0$, $(L_2)_2$, $(L_2)_4$, $(L_2)_6$ of even index removed, as in Figure 4(b).

Figure 5. Odd deletion: two enantiomorphic configurations, both with symbol $8\#(2,1;3,2;*1,3);D$. In both cases, points v_0 and lines L_0 are blue.

Figure 5(a) has lines labelled L_2 of odd index removed (that is, lines $(L_2)_1$, $(L_2)_3$, $(L_2)_5$, $(L_2)_7$ were removed), while Figure 5(b) has lines labelled L_2 of even index removed (that is, $(L_2)_0$, $(L_2)_2$, $(L_2)_4$, $(L_2)_6$ were removed).

We can trace the parity changes through this example. By convention, the points v_0 are type D. Since the first symbol $[[s_1, t_1]] = [[2, 1]] = v_1$ has entries of opposite parity, points v_1 are type MD. Since $[[s_2, t_2]] = [[3, 2]] = v_2$ again has entries of opposite parity, the points v_2 are type D. Since $[[s_3, t_3]] = [[1, 3]]$ has entries of the same parity, the points v_0 are (still!) type D. The symbol sequence $*1, 3$ indicates that half of the third class of lines is removed; because $v_0 = [[s_3, t_3]] = [[1, 3]]$ are points of type D, diameters must be added. Finally, note that diameters pass through the points v_0 and v_2, so they contain four points each, and each of the points labelled v_0 has passing through it two lines labelled L_0, one line labelled L_2, and one diameter (and points v_2 similarly have four lines passing through them), so the result is indeed a 4-configuration.

In general, care must be taken to ensure that if diameters are added, there are only two classes of points of type D (although there may be as many classes as you like of points of type MD), and vice versa if mid-diameters are added. The following result determines allowable symbol sequences where there are two classes of points of one type and the rest of the other type.

Theorem 3.1. *(Odd deletion) Suppose for $h \geq 3$ the configuration*

$$m\#(s_1, t_1; s_2, t_2; \ldots ; s_h, t_h)$$

has the following properties:

1. *s_h and t_h, the last two terms in the configuration symbol, are both odd;*
2. *the pairs s_{h-1}, t_{h-1} and s_1, t_1 are of opposite parity;*
3. *all other pairs s_j, t_j for $2 \leq j \leq h-2$ are of the same parity.*

Then half the lines labelled L_{h-1} may be removed and diameters may be added to form an (n_4) configuration.

Note that the requirement that $h \geq 3$ is not arbitrary; there are no 2-celestial configurations whose last two entries are both odd.

Proof. The lines labelled L_{h-1} contain points labelled v_{h-1} and $v_h = v_0$. Since s_h and t_h are both odd, removing half of the lines labelled L_{h-1} results in each point labelled v_0 and v_{h-1} containing a single line labelled L_{h-1} By construction, points labelled v_0 are type D, so points v_{h-1} are also type D. Since s_1 and t_1 are of opposite parity, points v_0 and v_1 are of the opposite type, so points v_1 are type MD. Since s_j and t_j are of the same parity for $2 \leq j \leq h-2$, all the points labeled $v_2, \ldots, v_{h-2}$ are of the same type as v_1 and thus are also of type MD. Finally, since s_{h-1} and t_{h-1} are of opposite parity, the points v_{h-1} switch type and are of type D. Therefore, the only points of type D are v_0 and v_{h-1}, while the rest are of type MD; adding diameters yields an (n_4) configuration. □

For clarity, we present parity patterns for configuration symbols for small values of h. The notation E means that the entry is even, O means the entry is odd, X, X in a pair means the pair must be of the same parity, and X, Y in a pair means the pair must be of the opposite parity.

h = 3: $m\#(X, Y; X, Y; *O, O); D$
h = 4: $m\#(X, Y; X, X; X, Y; *O, O); D$
h ≥ 5: $m\#(X, Y; \underbrace{X, X; \ldots; X, X;}_{h-3} X, Y; *O, O); D$

In particular, there are several trivial families of configurations formed by odd deletion, including:

- $m\#(a, b; c, a; *b, c); D$ where b and c are odd and a is even (Figure 5);
- $m\#(a, b; c, d; b, a; *d, c); D$ where c and d are odd and a and b are of opposite parity;
- $m\#(a, b; c, d; e, a; b, c; *d, e); D$ where b is even and a, c, d and e are all odd (Figure 6).

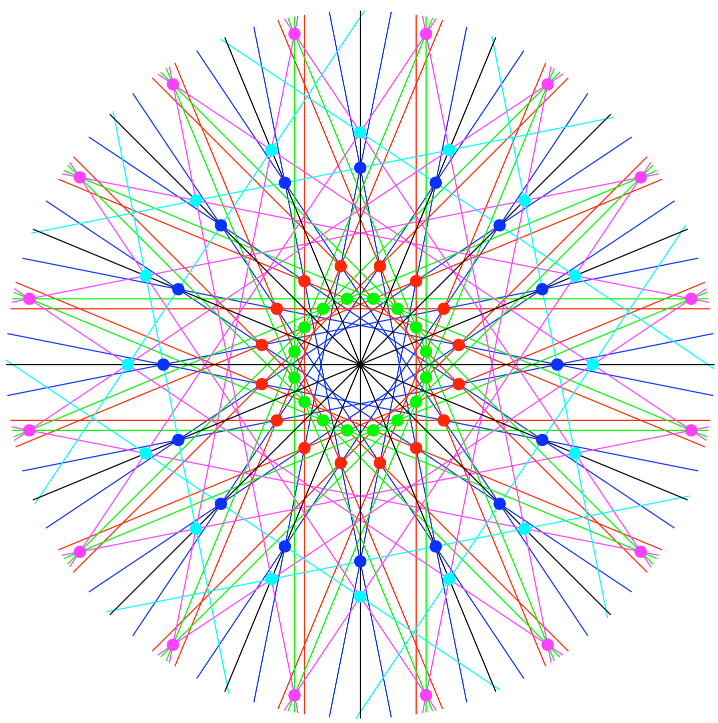


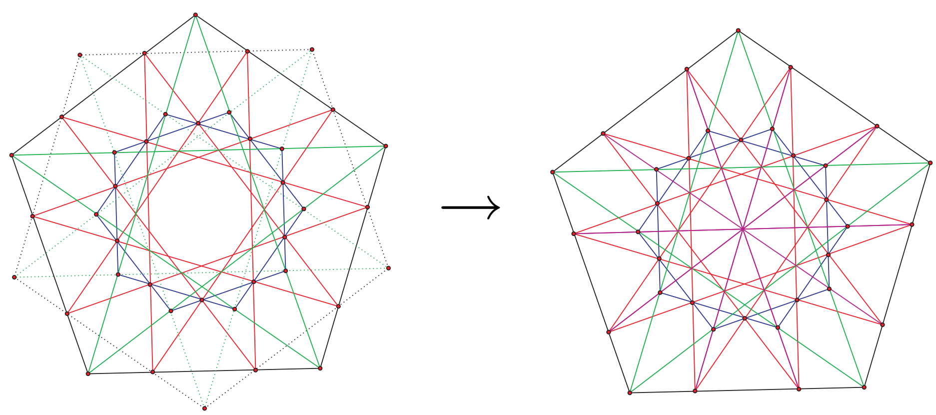
FIGURE 6. The configuration

$$16\#(7, 6; 5, 3; 1, 7; 6, 5; *3, 1); D.$$

The points v_0 and lines L_0 are shown in blue; half of the lines L_4, in cyan, have been deleted.

4. Deleting points and lines and adding mid-diameters: $(hm_4) \to \left(\left((h-1)m + \frac{m}{2}\right)_4\right)$

In the previous construction technique, half of a symmetry class of lines was deleted and diameters were added, but it was not necessary to delete any points and (usually) the number of points and lines in the configuration remained constant. In the following construction technique, half of a symmetry class of points and of lines will be deleted and mid-diameters will be added, yielding a configuration with a smaller number of points and lines. An example is shown in Figure 7.



$10\#(2,1;4,3;1,2;3,4)$ $\longrightarrow$ $10\#(*2,1;4,3;1,2;*3,*4); MD$

Figure 7. Even deletion. The points v_0 are the outermost ring of points; lines L_0 are black and lines L_3 are green.

Theorem 4.1. *(Even deletion) Suppose for $h \geq 3$ that*

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_h, t_h)$$

has the following properties:

1. *t_h and s_1 are distinct, and both are even (that is, the first and last entries in the symbol are even);*
2. *s_h and t_1 are odd (the second and second-to-last entries);*
3. *If $h > 3$, the pairs s_2, t_2 and s_{h-1}, t_{h-1} are of opposite parity, while if $h = 3$, the pair s_2, t_2 is of the same parity;*
4. *All other pairs s_j, t_j, for $3 \leq j \leq h-2$, are of the same parity.*

If every other point labelled v_0 and all the lines incident with those points are removed—that is, half the symmetry class of lines labelled L_0 and half the symmetry class labelled L_{h-1}—then mid-diameters, which will pass through points labelled v_1 and v_{h-1}, may be added to form a $\left(\left((h-1)m + \frac{m}{2}\right)_4\right)$ configuration.

Again, the requirement that $h \geq 3$ is not arbitrary; although there are 2-celestial configurations whose first and last entries are even and second and second-to-last entries are odd, there are too few rings of points for the construction to work.

Proof. Recall that lines L_0 contain points labelled v_0 and v_1 and lines L_{h-1} contain points labelled v_{h-1} and v_0. Since t_h and s_1 are both even, when we remove half the points labelled v_0, half of the lines labelled L_{h-1} contain two points labelled v_0 and the remainder contain none (by considering t_h) and likewise, considering s_1, every other line L_0 contains two points labelled v_0 and the rest none. The theorem instructed us to remove the lines which now contain no points labelled v_0 (since before removal, these were the very lines incident with the removed points).

Now consider the lines L_0, half of which have been removed. These lines also contain points labelled v_1, and in fact the lines labelled L_0 are of span t_1 with respect to these points. Since t_1 is odd, when half of the lines L_0 are deleted, each point labelled v_1 has one line labelled L_0 passing through it. Similarly, since the lines L_{h-1} are of span s_h with respect to the points v_{h-1} and s_h is odd, when half of the lines L_{h-1} are removed, each point v_{h-1} has one line labelled L_{h-1} passing through it.

We need to show that the points v_1 and v_{h-1} are the only points of type *MD* in the configuration, so that when we add mid-diameters, only four points will be incident with those mid-diameters. To do this we will apply Lemma 2.1 multiple times. By definition, points v_0 are of type D. Since s_1 and t_1 are opposite parity, points v_1 are of type *MD*. Suppose that $h \geq 4$; in this case, by hypothesis, s_2 and t_2 are also of opposite parity, so points v_2 are of opposite type, that is, type D. Now, the theorem says all pairs s_j, t_j for $3 \leq j \leq h-2$ are of the same parity, so all points $v_3, \ldots, v_{h-2}$ are of type D. Finally, since pairs s_{h-1}, t_{h-1} and s_h, t_h are of opposite parity, points labelled v_{h-1} and $v_h = v_0$ are of type *MD* and D, respectively. In the case where $h = 3$, the situation is even more straightforward: we begin with points v_0 of type D. Since s_1, t_1 are of opposite parity, points v_1 are type *MD*. Since s_2, t_2 are the *same* parity, points v_2 are also type *MD*. Finally, the opposite parity of s_3, t_3 switches $v_3 = v_0$ back to type D.

In particular, only points labelled v_1 and v_{h-1} are of type *MD*; conveniently, these were the very points which, after deletion of half of the lines labelled L_1, had only three lines passing through them. Adding in the mid-diameters contributes a fourth line through each point, and each mid-diameter passes through two points labelled v_1 and two labelled v_{h-1}.

The total number of points has been reduced by $\frac{m}{2}$ as has the total number of lines, since m lines were deleted and $\frac{m}{2}$ mid-diameters were added. □

Again, for clarity, especially since the patterns differ slightly for small numbers of rings, we present the parity patterns needed for $h = 3, 4$ and $h \geq 5$.

h = 3: $m\#(*E, O; X, X; *O, *E); MD$
h = 4: $m\#(*E, O; X, Y; X, Y; *O, *E); MD$
h ≥ 5: $m\#(*E, O; X, Y; \underbrace{X, X; \ldots; X, X;}_{h-4} X, Y; *O, *E); MD$

There are several trivial families of configurations formed by even deletion (that is, families beginning with trivial celestial configurations), including:

- $m\#(*a, b; c, a; *b, *c); MD$ for a and c even and b odd (Figures 8(a) and 8(c));
- $m\#(*a, b; c, d; b, a; *d, *c); MD$ for a, c even and b, d odd (Figure 7);
- $m\#(*a, b; c, d; e, a; b, c; *d, *e); MD$ for a, c, e even and b, d odd (Figure 9).

(Note that while Figure 8(b) is also formed by applying even deletion to a 3-celestial configuration, the original 3-celestial configuration is not trivial.)

Figure 7 shows a four ring configuration formed using even deletion. In the figure shown on the left hand side of Figure 7, half of the points labelled v_0 are to be deleted, along with half of the lines labelled L_0 and L_3 (these are shown dashed); on the right hand side, the points and lines have been deleted, and mid-diameters have been added. Figures 8 and 9 show examples of three and five ring configurations, respectively, formed by even deletion.

This construction method is especially useful for constructing small configurations; three of the known symmetric (25_4) configurations may be constructed in this way, beginning from celestial (30_4) configurations. They are shown in Figure 8.

5. Combining deletion techniques

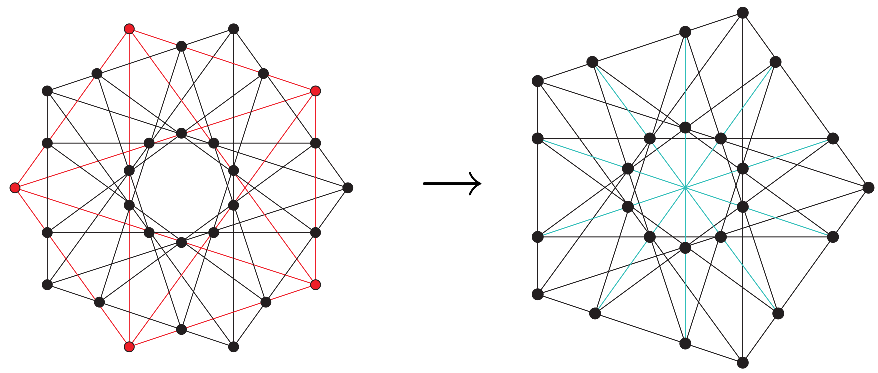
Given an appropriate symbol for a celestial configuration, the two different types of deletion methods may be combined to produce other symmetric 4-configurations.

In a celestial configuration $m\#(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)$ where pairs s_i, t_i, for i even, are both odd and pairs with odd i are of opposite parity, half of the line classes L_1 and L_3 may be removed and both diameters and mid-diameters may be added to form the configuration

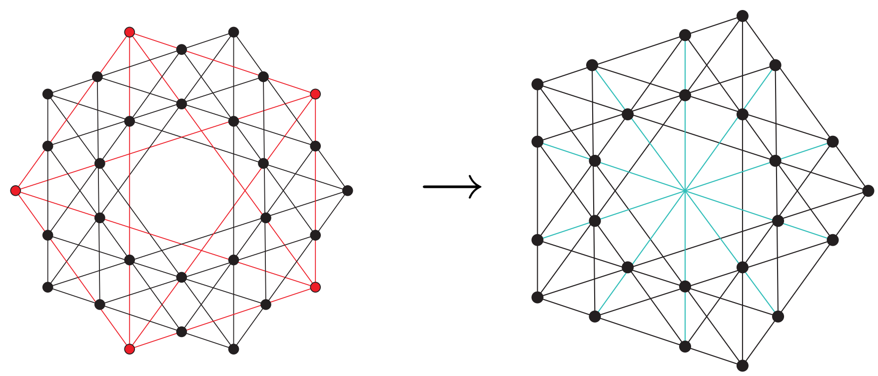
$$m\#(s_1, t_1; *s_2, t_2; s_3, t_3; *s_4, t_4); D; MD\,.$$

That is, a slight generalization of the odd deletion technique has been applied twice to the same configuration. Figure 10 shows the configuration $10\#(1, 2; *3, 5; 2, 1; *5, 3); D; MD$.

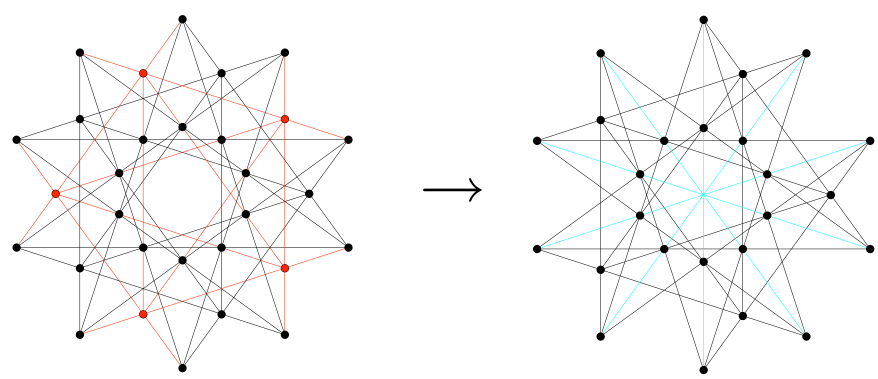
Both even and odd deletions are combined in Figure 11, which shows a (42_4) configuration with symbol $14\#(*6, 4; *2, *3; 4, 6; *3, *2); MD$. It has had half of the lines labelled L_0 (blue) and L_3 (magenta) removed, along with half the points labelled v_0 (blue), and also half the points labelled v_1 (red) and lines labelled L_1 (red) removed, and mid-diameters added.



(a) $10\#(*4,1;2,4;*1,*2)$; *MD*.



(b) $10\#(*4,3;1,3;*1,*2)$; *MD*.



(c) $10\#(*4,3;2,4;*3,*2)$; *MD*.

FIGURE 8. Three of the known symmetric (25_4) configurations, all formed by even deletion. In (a) and (b), the points v_0 form the outermost ring of points, while in (c), the points v_0 are the middle ring of points.

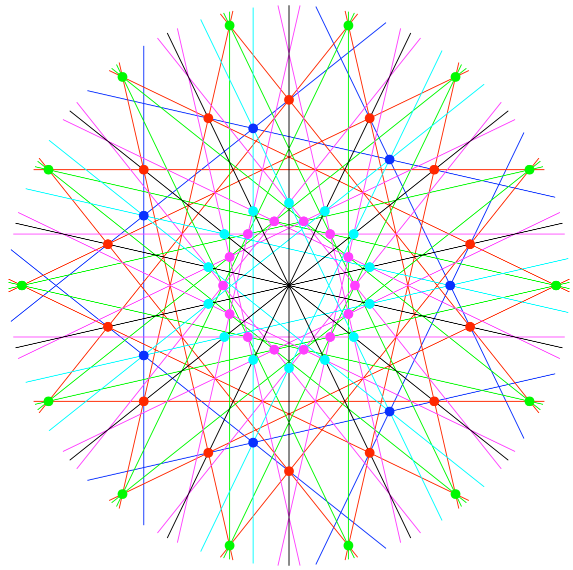


FIGURE 9. More even deletion:

$$14\#(*2,3;4,5;6,2;3,4;*5,*6).$$

The points v_0 and the lines L_0 are shown in dark blue.

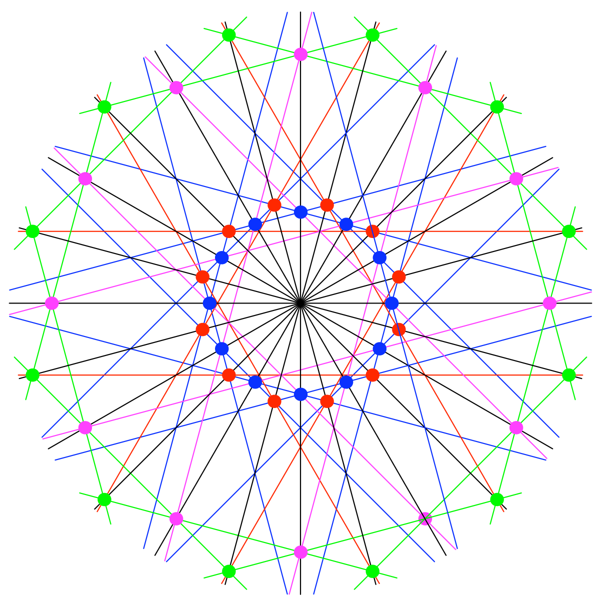


FIGURE 10. The configuration

$$12\#(1,2;*3,5;2,1;*5,3);D;MD,$$

which has half of two symmetry classes of lines removed and diameters and mid-diameters added.

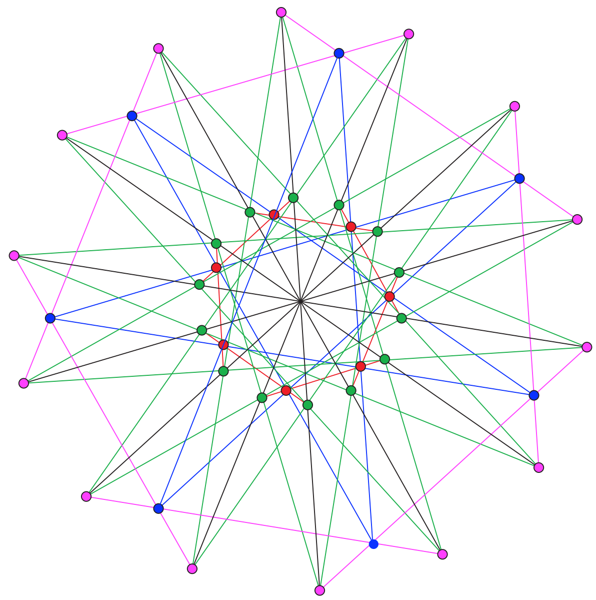


FIGURE 11. Multiple modifications; a (42_4) configuration with symbol $14\#(*6,4;*2,*3;4,6;*3,*2);MD$. Points v_0 and lines L_0 are blue.

6. Open Questions

Undoubtedly, there are many more variants of deletion techniques to be found.

For example, the odd and even deletion constructions do not require beginning with a celestial configuration—celestial configurations simply provide a convenient class of examples to work with. Similar constructions are possible with other classes of 4-configurations. For example, starting from the "floral configuration" (see [3] for details on this kind of configuration) in Figure 12(a), by deleting the lines shown in red and replacing them with the green lines shown in Figure 12(b), we obtain a new (72_4) configuration. Analogous deletion constructions can be carried out with several other types of floral configurations. In the sequel to this paper we shall investigate a number of other deletion constructions.

The deletion constructions presented here are based on eliminating every other point in a ring of points. Are there analogous constructions where every third point is eliminated? Every fourth point?

Are there interesting configurations which can be constructed using deletion techniques or combinations of deletion techniques that do not lead to (n_4) configurations, but rather to configurations with higher numbers of incidences of points and lines (e.g., (p_4, n_5) configurations?).

What configurations may be constructed beginning with other classes of configurations, such as 3-configurations or 6-configurations?

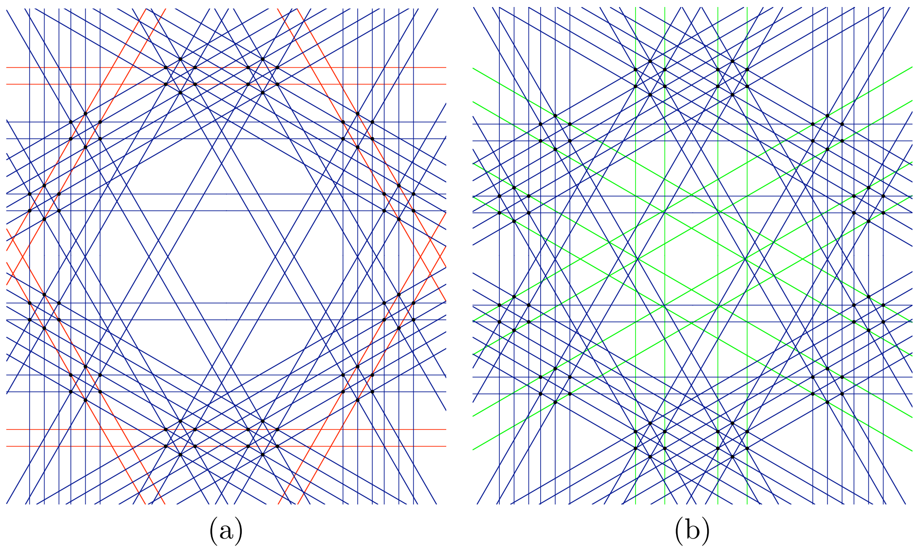


FIGURE 12. (a) A (72_4) floral configuration. If the red lines are deleted and replaced with the green lines shown in (b), we obtain a new (72_4) configuration.

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DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF ALASKA FAIRBANKS, FAIRBANKS, ALASKA, USA
E-mail address: lwberman@alaska.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON, USA
E-mail address: grunbaum@math.washington.edu