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[4,3]-Configurations with Many Symmetries

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Dedicated to memory of my admired teacher, Aryeh Dvoretzky, who passed away earlier today (May 8, 2008)

A family of points and lines in the Euclidean or projective plane is called a [q,k]-configuration provided each of the points is incident with precisely q of the lines, and each line with precisely k of the points. If there are p points and n lines, we say that it is a (p_q,n_k) configuration. If q = k then obviously p = n, and we talk about a k-configuration or an (n_k) configuration.

3-configurations have been studied for well over a century, and 4-configuration have received increasing attention over the last 20 years – in GEOMBINATORICS and other journals. Scant attention has been given in the literature to [q,k]-configurations with $q \neq k$. Some references will be given at the end of this note.

The main purpose here is to present a method of construction of [4,3]-configurations that appears to be new. The interest in the construction arises from the following fact. Various [3,4]-configurations have been described and are easy to find – all that is needed is the set of vertices of two suitable concentric regular polygons, with some of their diagonal lines. See Figure 1. However, all these and other known examples with considerable symmetry have lines going through the "center" of the configuration. Hence, if one tries to construct [4,3]-configurations by taking images of the [3,4]-configurations under polar reciprocation, one winds up either with configurations lacking the symmetry (if the center of the reciprocating circle does not coincide with the center of the configuration), or with configurations that involve "ideal points" (points at infinity). Most other constructions yield only asymmetric configurations, see Figure 2. Our construction of (p_4,n_3) configurations yields them directly, apparently for all feasible values of $p \geq 15$. Examples are shown in Figures 5, 6 and 7.

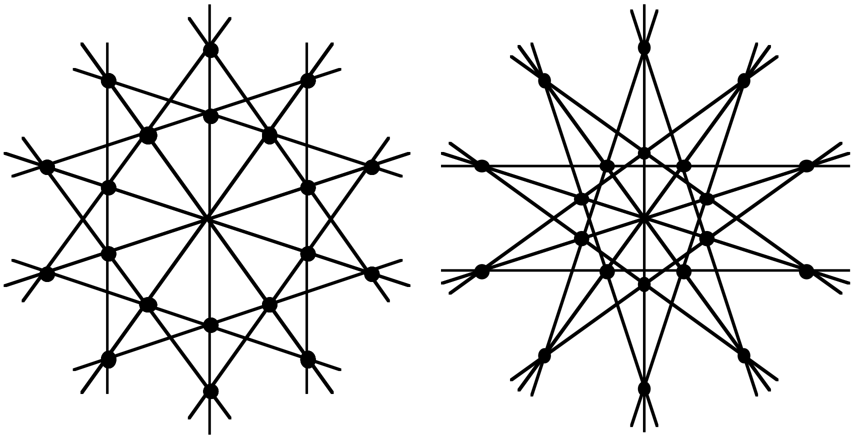


Figure 1. Two examples of configurations $(20_3,15_4)$ with a high degree of symmetry.

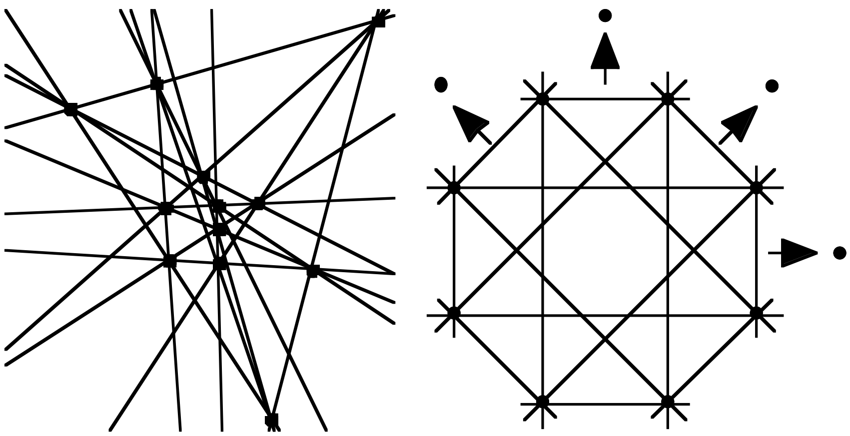


Figure 2. Two $(12_4,16_3)$ configurations. The one at left lacks symmetry, the other has "points at infinity" in the extended Euclidean plane.

We note that purely combinatorial considerations show that necessary conditions for the existence of a (p_4,n_3) configuration are $4p = 3n$, $p \geq 0$, $n \geq 10$. It follows that the only possible configurations must have the form $((3r)_4,(4r)_3)$, for $r \geq 3$. It is well known (and not hard to prove) that the case $r = 3$ can be realized only combinatorially, and not with geometric configurations we consider here. For r = 4, that is, for configurations $(12_4,18_3)$ and $(18_3,12_4)$ various examples are known, many based on families of points on cubic curves. An example of the former appears in Figure 3. For larger values of r our construction yields the first published images of the [4,3]-configurations. Clearly, polarity produces analogous images of the [3,4]-configurations, which we do not show.

To explain the construction we need to recall the "polycyclic configurations" (n_3) described by Boben and Pisanski [3], in particular the tricyclic ones. The tricyclic configurations depend on several discrete parameters and one real-valued parameter; the discrete parameters are used in the symbol m#(b,c,d;e) for a family of configurations that are mutually isomorphic (have the same incidences), and the continuous parameter t_0 distinguishes between the various members of that family. We explain the construction of the tricyclic configurations $((3m)_3)$ by using the example of the configuration we denote 6#(2,1,2;4), with t_0 = 0.526588; it is illustrated in Figure 4.

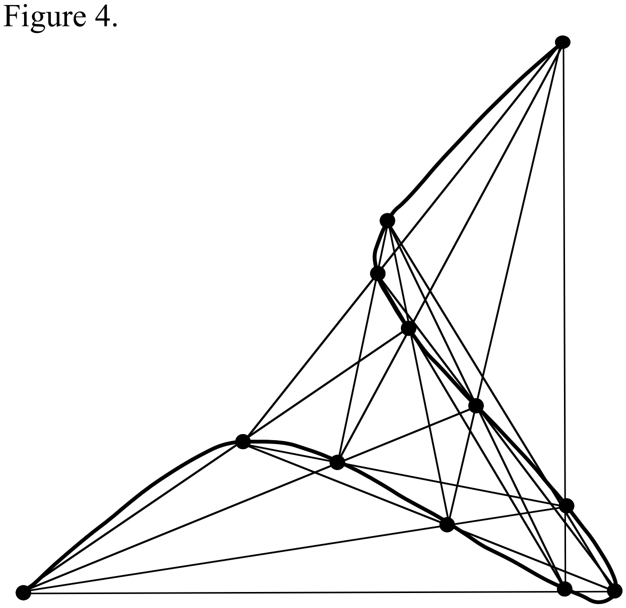
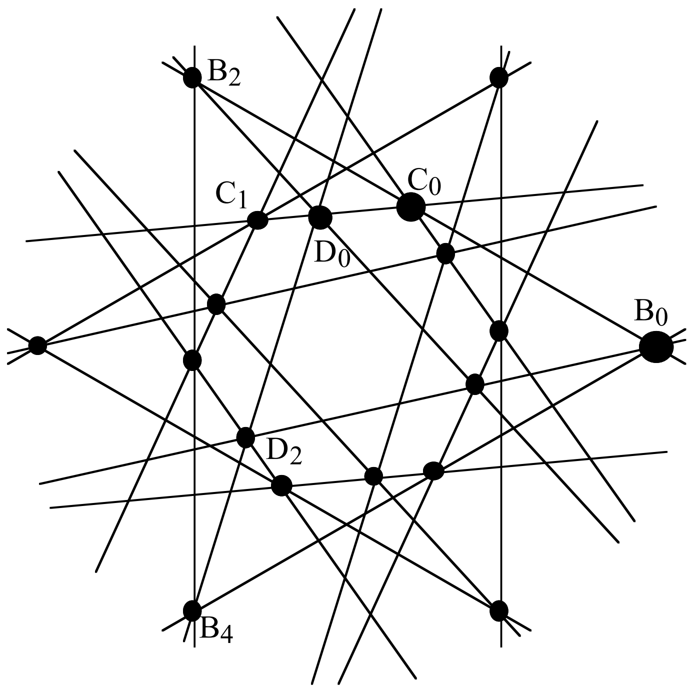


Figure 3. A$(12_4,\ 16_3)$ configuration, with points on a cubic curve. The equation of the curve, and the coordinates of the points, are available from the author. Adapted from Metelka [7].

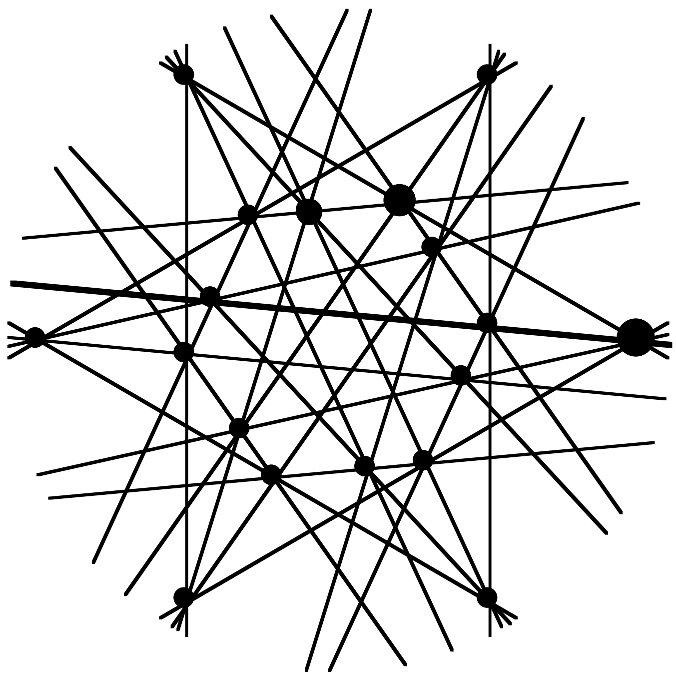


6#(2,1,2;4) $t_0 = 0.526588$; $t_1 = 0.61252$

Figure 4. An illustration of the construction of tricyclic configurations (n_3), exemplified by the configuration m#(b,c,d;e) = 6#(2,1,2;4). The meaning of the parameters is explained in the text.

We start by selecting the vertices of a regular convex m-gon, and denote them by B_0, B_1, ...B_{m-1}, in the counterclockwise direction. The points B_0 and B_b determine a line L_0 of the configuration, and similarly B_j and B_{j+b} determine a line L_j for all j. On L_0 we locate the point C_0 such that the ratio B_0C_0/B_0B_b is t_0, and similarly for the points C_j on L_j. On the line through C_j and C_{j+c} we choose a point D_j such that $C_jD_j/C_jC_{j+c} = t_1$, where t_1 is a number still to be determined. The line through D_j and D_{j+d} should pass through the point B_{j+e}. Carrying through the algebra, we find that the other parameters generate a quadratic equation for t_1; it can have two, one or no real solutions. Each such solution determines a tricyclic configuration. The value of t_1 depends on the choice of t_0 and the geometry of the configuration, even though the incidences are the same. This is crucial to our construction. In Figure 4, as in later ones, the points B_0, C_0, D_0 are marked by larger dots than the remaining points, with decreasing size of the three.

The idea of the construction of the [4,3]-configurations is to start with a tricyclic m#(b,c,d;e) configuration, and choose t_0 such that (with the resulting t_1) B_0 becomes collinear with C_x and D_y, for appropriate integers x and y. For the configuration 6#(2,1,2;4) in Figure 4 this is illustrated in Figure 5; the t_0 value in the former is the one that yields the $(18_4,24_3)$ configuration in the latter. In general, for any given m#(b,c,d;e), there may be several (or no) suitable values of the parameters t_0 and t_1, as well as x and y.



6#(2,1,2;4)
$t_0 = 0.526588$; $t_1 = 0.61252$; $x = 5$; $y = 1$

Figure 5. A [4,3]-configuration $(18_4,24_3)$ obtained from the tricyclic configuration (18_3) in Figure 4. The fourth line (through B_0, C_5 and D_1) is emphasized.

In Figures 6 and 7 we show additional examples of configuration $(15_4,20_3)$ and $(21_4,28_3)$ obtained by our construction.

It would be very desirable to determine all parameters m, b, c, d, e, t_0, t_1, that yield [4,3]-configurations, or at least to characterize the discrete parameters that result in such configurations. However, we are unable to do that, for a variety of reasons.

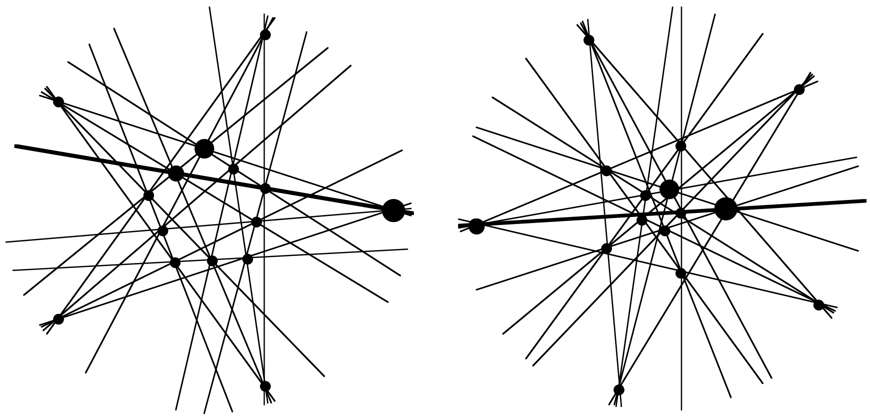


Figure 6. Two $(15_4,20_3)$ configurations, both 5#(2,1,3;2) with x = 4 and y = 0. For the first $t_0 = 0.562796$, $t_1 = 0.51946$; for the second $t_0 = 0.477389$, $t_1 = 8.01928$.

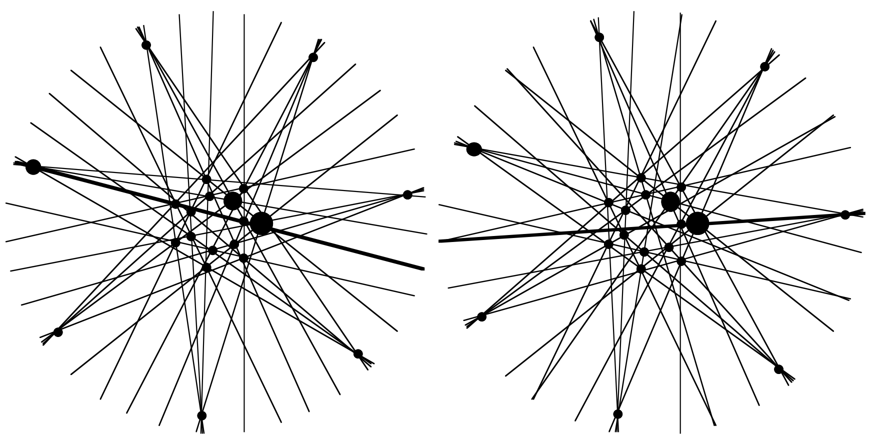


Figure 7. Two $(21_4,28_3)$ configurations, both 7#(2,1,3;5) with y = 0. For the first $t_0 = 0.521921$, $t_1 = 8.3525$, x = 3; for the second $t_0 = 0.484401$, $t_1 = 8.07463$, x = 6.

The main difficulty is that the discrete parameters appear in the crucial algebraic equations for the t_i's in coefficients that involve trigonometric functions in complicated ways, so that neither computer algebra nor any manual tricks could give explicit relations. Naturally, for any given selection of the discrete parameters, it is easy to obtain numerical solutions; this is how the diagrams in this note were drawn.

Extensive experimentation leads to the conviction that for all $m \geq 5$, the choice (b,c,d) = (2,2,2) can be paired with suitable values of e and the t_i's. However, even this remains only a conjecture. The situation is similar for many other selections of (b,c,d), but there is not enough numerical evidence to conjecture a characterization of such parameters, or of the values of e that may be used with them. An example of a larger [4,3]-configuration is shown in Figure 8.

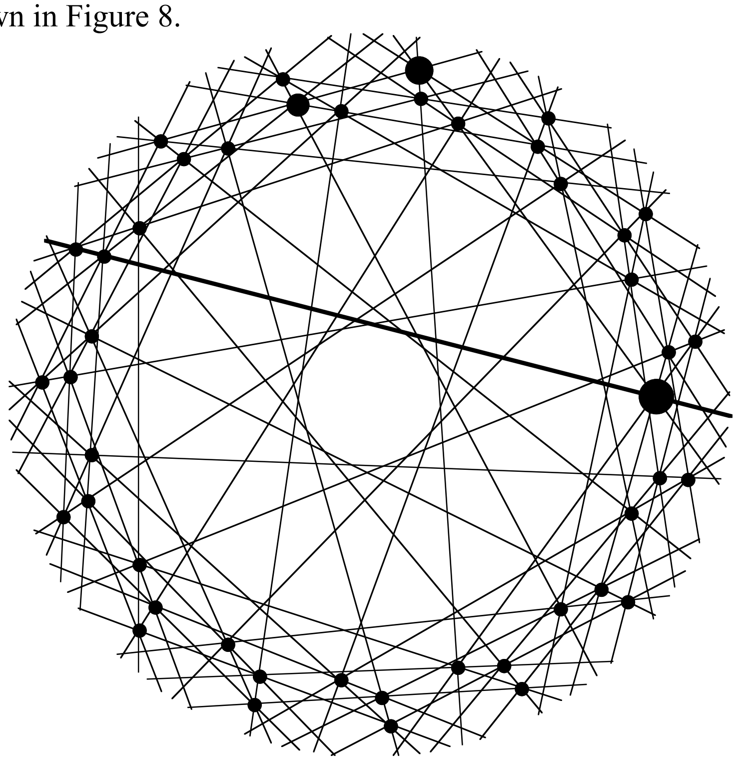


Figure 8. A $(45_4,60_3)$ configuration 15#(3,2,2;6), with x = 3 and y = 2.

The only geometric [4,3]- and [3,4]-configurations that have been extensively investigated are $(12_4,16_3)$ and their duals. Many references to the corresponding papers can be found in [3] and [4], where it is also stated that there are exactly 574 *combinatorial* configurations $(12_4,16_3)$. However, the number of *geometrically* realizable ones is not known; the investigation of that question is complicated by the fact that in several of the publications it is not clear whether the plane they consider is the Euclidean or the complex plane. As mentioned earlier, in the literature there is no consideration of [4,3]-configurations with $n \geq 15$ and with any reasonable symmetry in the Euclidean plane.

Some geometric [q,k]-configurations with $q \geq 4$, $k \geq 4$, $q \neq k$, are discussed in [1], [2] and [5].

References.

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