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Configurations of points and lines.

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Configurations of Points and Lines

Branko Grünbaum

1. Introduction

Configurations — as the word is interpreted throughout this paper — are simple enough that in their geometric aspect could be explained to any third-grader, but easily lead to problems that are beyond the reach of all presently available tools. While one could reasonably place configurations within the purview of elementary geometry, they could also be interpreted as belonging to algebraic geometry, or combinatorics, or topology. Despite — or possibly because — the simplicity of the concept, over the years it suffered from many instances of confusion and downright errors. Even if this fact may be understood as arising from misinterpretation of the nature of configurations, it is surprising that the problematic statements were in many cases not recognized and corrected for nearly a century.

Configurations of points and lines were first defined by Reye [**R1**] some 125 years ago (see also [**R2**]). Many specific configurations have been described and studied before that — but without a framework into which they could fit. Configurations enjoyed considerable popularity during the last two decades of the nineteenth century, only to be relegated to a mathematical limbo afterwards. Neither the publication of the only serious text on configurations by Levi [**L2**] in 1929, nor the attempt by Hilbert and Cohn-Vossen [**H1**] in the 1930's, generated much activity. In the second half of the twentieth century there was a slight renewal of interest, in particular through Coxeter's frequently quoted paper [**C3**]. This, and other papers of Coxeter's on configurations make the topic appear appropriate for the present volume, as it was for the conference that led to it.

During the last two decades, research on configurations became energized by several factors.

- Re-examination of the nineteenth century papers revealed that many of the basic results that have been accepted for a century or more are, in fact, not valid as stated. Details of this aspect will be presented in Section 3.
- Novel ideas made it possible to study various kinds of configurations that were beyond the reach of earlier investigations.
- At the same time, computers and computer graphics contributed to the appeal of the study, by enabling enumerations and depictions of a wide variety of configurations to an extent not possible earlier.

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These aspects will be discussed in the remaining parts of this survey. A large number of intriguing open problems will be presented, along with precise definitions and statements of results, and with many references to the scattered literature.

Following general definitions in Section 2, Sections 3 and 4 deal with the history of the early papers on configurations. These sections describe in some detail the errors of these basic papers. It seems appropriate to dwell on the shortcomings as a warning: Very capable people were capable of making mistakes, and the lack of clarity and precision causing them has been known to occur more recently as well. In later sections we give brief explanations of recent developments in the theory of configurations. Through this it becomes apparent that as soon as the right questions are asked, there is no more room for dismissal of configurations as trivial geometry.

2. Definitions and notation

DEFINITION 2.1. A *configuration* $\mathcal{C}$ is a family of *points* (or *vertices*) and *lines* such that for some positive integers p, q, n, k each of the p points is incident with q of the n lines, and each of these lines is incident with k of the points. Such a configuration $\mathcal{C}$ is denoted (p_q, n_k) or, if $p = n$ and hence $q = k$, more simply by (n_k). If the parameters p and n are not relevant, a configuration (p_q, n_k) is said to be of *type* $[q, k]$.

Several explanations and clarifications should help understand the scope of this definition. To begin with, the points and lines are understood to be part of some space in which these concepts, and incidence, are meaningful. Our main settings are the real *projective* plane and the real *Euclidean* plane E^2; this is the meaning unless some other interpretation is specifically mentioned. We shall invariably interpret the projective plane as the *extended Euclidean plane* E^{2+}, that is, E^2 augmented by the "ideal points", also called "points at infinity". For geometric configurations "point" and "line" mean a point of the plane and a straight line, respectively. "Incidence" is defined as the point belonging to the line; however, we allow that some points of the configuration lie on a line with which they are not considered to be incident. To avoid trivialities we shall always assume that not all points of a geometric configuration are collinear.

In a different interpretation of Definition 2.1 we obtain combinatorial configurations: here "points" are understood as any symbols, "lines" as sets of "points", and "incidence" as "point is an element of a line". Thus, combinatorial configurations are special incidence structures. Combinatorial configurations are often specified by a configuration table, in which columns represent points incident with one line. In our considerations we insist that in a combinatorial configuration two distinct points are incident with at most one line; this implies that two distinct lines are incident with at most one point.

The earliest mention of the fact that for every choice of q and k there exist geometric configurations of type $[q, k]$ seems to be in the paper of Kantor [**K2**] published in 1879. His construction is quite complicated. As is easily seen, the existence of such configurations is more easily established by considering a block of size k^q of points in the integer lattice of the q-dimensional Euclidean space, and the lines through them parallel to the coordinate axes.

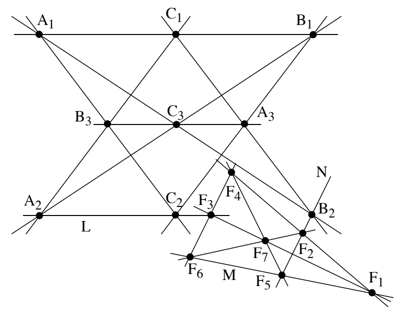


FIGURE 1. The combinatorial configuration (16_3) indicated by this diagram has a representation as shown, but it admits no realization since by the theorem of Pappus the point B_2 necessarily lies on the line L.

DEFINITION 2.2. Configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be *isomorphic* if there is a bijection between their points, and a bijection between their lines, which preserve incidences. If there are incidence-preserving bijections between points of one configuration and lines of the other, the configurations are said to be *dual* to each other. A configuration is *selfdual* if it is isomorphic to its dual.

Clearly, every geometric configuration is isomorphic to a combinatorial one; however, as we shall see, the converse is not valid.

Discussions of the relation between geometric configurations and combinatorial ones requires a differentiation which seems largely absent in the literature. A geometric configuration $\mathcal{C}$ is a *realization* of the underlying combinatorial configuration if a point and a line of $\mathcal{C}$ are incident *if and only if* their combinatorial counterparts are incident. In contrast, $\mathcal{C}$ is a *representation* of the underlying combinatorial configuration provided a point and a line of $\mathcal{C}$ are incident *whenever* their combinatorial counterparts are incident, but points may belong to other lines as well. The essential distinction between representation and realization is illustrated by Figure 1. Since every combinatorial configuration could be represented by a collinear set of points, by convention representations by such configurations and their duals are excluded from all following considerations and statements.

Levi graphs of configurations (and more general incidence structures) are one tool that simplifies discussions and leads to parsimonious ways of defining concepts. They were introduced by Levi [**L3**] in 1942, but were brought to the attention of a wider audience only by the path-breaking paper of Coxeter [**C3**]. The Levi graph $L(\mathcal{C})$ of a configuration $\mathcal{C}$ is a bipartite graph in which the vertices of one part correspond to the points of $\mathcal{C}$, and the vertices of the other part to the lines of $\mathcal{C}$; two vertices of $L(\mathcal{C})$ are connected by an edge if and only if the corresponding line and point are incident. As an illustration, in Figure 2 is shown the Levi graph of the configuration of Figure 1. The earlier "Menger graphs" of configurations discussed by Coxeter [**C3, C2**] are still used sometimes, but they are much less useful.

The Levi graphs of combinatorial configurations are easily characterized. For configurations (n_3) these are 3-valent bipartite graphs of girth (length of shortest circuit) at least 6, and similarly for other classes of combinatorial configurations. The graph-theoretic connection makes it redundant to dwell at length on definitions

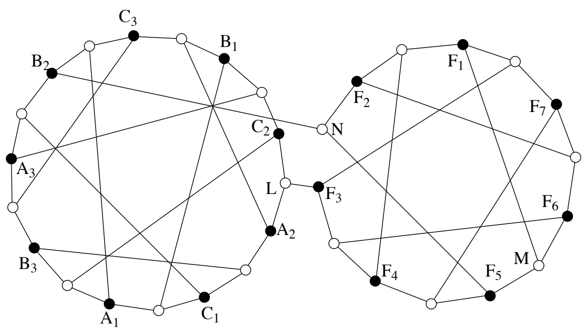


FIGURE 2. The Levi graph of the (16_3) configuration from Figure 1. (Adapted from the Levi graphs of Pappus and Fano configurations given by Coxeter [**C1**].)

of concepts such as isomorphism or duality, connected or k-connected configuration, Hamiltonian circuit in a configuration, or the girth of a configuration.

Another tool used in the study of configurations are *incidence matrices*, and various ways of representing them; see, for example, [**L2, B3**].

Neither of these tools leads to a solution of a central problem: *Which combinatorial configurations admit representations or realizations in the (Euclidean or projective) plane.* In Section 3 we shall mention coordinatization, one approach that can be used to such an effect, but which has to be applied to individual configurations and is highly laborious unless the configurations are very small.

Symmetry is a central concept for the newer development of configurations. By this we understand an isometric map of the plane onto itself that maps a configuration onto itself. All symmetries of a configuration form its *symmetry group*. A geometric configuration is said to be of symmetry type $[h_1, h_2]$ provided its points form h_1 orbits, and its lines h_2 orbits under its symmetry group. We shall also say that such a configuration is $[h_1, h_2]$-*astral*. Clearly, if a configuration of type $[q, k]$ is $[h_1, h_2]$-astral, then $h_1 \geq (k+1)/2$ and $h_2 \geq (q+1)/2$. If h_1 and h_2 have these minimal values we shall simplify the language and say that the configuration is *astral*. In cases where $h_1 = h_2 = h$, we shall say that the configuration is h-*astral*. A variant of this term was introduced in [**G10**] along with a variety of examples. Here we shall see many examples in Sections 5, 6, and 7.

3. Early results and errors, and their subsequent development

Most of the early works on configurations deal with configurations (n_3), and in particular, with the enumeration of isomorphism classes for certain values of n. One of the startling facts about these investigations is how hard it is to decide nowadays whether they discuss combinatorial configurations or geometric ones, and in the latter case, whether they consider configurations in planes or spaces over the reals, or over complex numbers. All these possibilities were often mixed up — for example, in the writings of Kantor, Schönflies, Schroeter, Steinitz and others listed in the references. As we shall see soon, contributing to the confusion is the fact that results valid for some special cases were deemed to have general validity.

Other investigations concerned the possibility of geometric realizations of combinatorial configurations, and of the possibilities of constructing all combinatorial

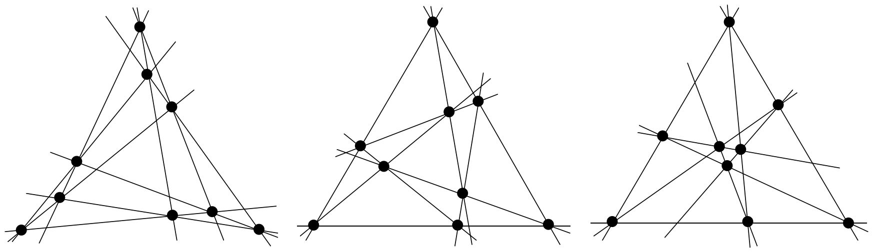


FIGURE 3. The three configurations (9_3). The notation is the one used by [**H1**]. It is not clear who first devised the realization with 3-fold rotational symmetry of these configurations; the ascription by Steinitz ([**S8**, p. 489], [**S10**, p. 158]) to a 1910 paper by H. A. Schwarz is mistaken.

configurations (n_3). It is unsettling to realize how many errors, small and large, were committed in these investigations. It is even more surprising that some of the errors remained undetected for a century.

3.1. The first enumeration of the isomorphism classes of configurations (8_3) and (9_3) was carried out by Kantor [**K3**]. Kantor explains that *the unique configuration (8_3) can be constructed by starting with a simple quadrangle, to which another quadrangle is inscribed in such a way that it is also circumscribed to the starting one.* The mystery of how this can be done is removed later in the paper when he mentions, in passing, that if the vertices of the first quadrangle are in the real plane, then the other four vertices are imaginary. This was in a different context shown much earlier, by Möbius [**M3**]. In the review of [**K3**] by Schubert [**S5**] the description italicized above of the configuration is repeated, but there is no mention of the reality or otherwise of that configuration. A proof of the non-realizability of the configuration (8_3) was given by Möbius [**M3**] (see Coxeter [**C3**, pp. 122, 131]), Levi [**L2**, p. 99], and by Bokowski and Sturmfels [**B9**, p. 35]. More information concerning the configuration (8_3), its realizability in various planes and its symmetries, together with historical remarks and references can be found in Coxeter's papers [**C3, C5**].

Kantor [**K3**] also established that there are precisely three non-isomorphic combinatorial configurations (9_3), and that each is isomorphic to a geometric configuration; he provides diagrams of all three. We show realizations of these configurations in Figure 3. One of them, the Pappus configuration $(9_3)_1$, is an expression of one of the basic theorems of projective geometry. Hilbert and Cohn-Vossen [**H1**] and Coxeter [**C3, C5**] provide illustrations and details about the realizations and symmetries of the Pappus configuration $(9_3)_1$.

Strangely enough, Kantor never mentions the unique combinatorial configuration (7_3); it is generally known as the Fano configuration, and is important in many combinatorial contexts. The omission may possibly be explained by the fact that this combinatorial configuration is not isomorphic to any geometric configuration in either the real or the complex plane.

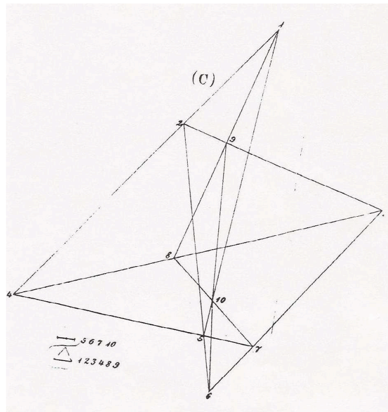


FIGURE 4. Facsimile of one of the diagrams of [**K3**], purporting to show a (10_3) configuration. This configuration has been shown by Schroeter [**S4**] to be not realizable in the Euclidean plane. In the diagram, the line 9-10-6 is clearly not straight.

A later paper of Kantor [**K4**] is devoted to the enumeration of (isomorphism classes) of configurations (10_3). The enumeration is based on a mixture of combinatorial and geometric arguments, and is quite opaque. One of its results — that *there are ten different classes of combinatorial configurations* (10_3) — is correct, although it relies on an invalid argument. This argument, made without any justification, asserts that two configurations (n_3) are isomorphic if and only if their families of remainde r figures contain the same figures in the same numbers. (The *remainder figure* of a point P in a configuration consists of those points of the configuration that are not on any configuration line through P, and of configuration lines containing two or more of these points.) This statement was also repeated in the review of [**K4**] by Rodenberg [**R4**], although it is false in this generality; while true for $n \leq 10$, already for $n = 11$ there are counterexamples. Independent enumerations of the combinatorial configurations (10_3) were carried out by Martinetti [**M1**], Schroeter [**S4**], and more recently by many others (for example, [**S11, B4**]). The enumeration by Zacharias [**Z2, Z3**] is not correct, nor is the report on it by Togliati [**T1**].

The other claim of [K4], that *each of the ten classes of combinatorial configurations* (10_3) *is realizable as a geometric configuration of points and lines in the real plane* — is invalid. Kantor claims to derive his assertion using a statement credited to Johann Benedict Listing (but without any reference) to the effect that — in present-day terminology — every combinatorial (n_3) configuration can be interpreted as having a Hamiltonian circuit. (Kantor speaks of an n-gon inscribed and circumscribed to itself.) But the assertion attributed to Listing (which I have been unable to locate in any work of Listing's) is invalid, not only in the stated generality but even under quite stringent restrictions. (We shall discuss Hamiltonian circuits in more detail in Sections 3 and 4.) Kantor supports his claim of realizability by diagrams, purporting to show the corresponding ten geometric configurations. However, several of the diagrams do not actually show all intersection points, and some are incorrect. A facsimile of Kantor's purported realization of one

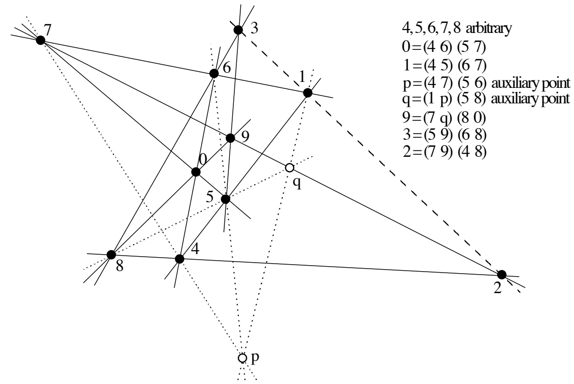


FIGURE 5. Schroeter's construction in [**S4**] of one of the (10_3) configurations.

of these configurations is reproduced in Figure 4; as clearly visible, the line 6,9,10 is not straight. But beyond inaccuracies of the diagrams, the important fact is that, as proved by Schroeter [**S4**], *one of the ten combinatorial configurations cannot be realized by points and lines in the real plane*; this configuration is the one shown in Figure 4. (Later investigations by Laufer [**L1**] and Bokowski-Sturmfels [**B9**] showed that this configuration cannot be realized by points and lines in any plane over a field, hence, in particular, not even in the complex plane.) In continuation of his paper [**S4**] Schroeter shows that the other nine combinatorial configurations (10_3) can be geometrically realized in the *real* plane. An example of his mode of construction is shown in Figure 5. In fact, the constructions Schroeter presents show that they are realizable in the *rational* plane as well. This has been confirmed using more modern methods by Bokowski-Sturmfels [**B9**].

The claim that every combinatorial configuration (n_3) can be realized by points and straight lines (in the real or complex plane) is repeated by Martinetti [**M1**], in the preliminaries to his enumeration of configurations (11_3). He makes that statement (which is repeated in the review of [**M1**] by Loria and Lampe [**L4**]) in order to claim that he enumerates geometric configurations while his argumentation deals only with combinatorial ones. What Martinetti finds is that there are 31 combinatorial configurations (11_3). (We shall return soon to another part of his paper [**M1**].) The same 31 configurations were independently found by Daublebsky [**D1**] using a different method. Going beyond this, in a supplement to [**D2**], Daublebsky presents diagrams showing that all these configurations are geometric. Without giving details, Daublebsky states that the diagrams were all obtained by the method used by Schroeter in [**S4**] in the construction of the nine realizable configurations (10_3), and that they are faithful and correct. An independent proof of the geometric realizability in the Euclidean plane of the 31 configurations (11_3) was provided by Sturmfels and White [**S12, S13**]; in fact, they establish the realizability of all 31 in the *rational* plane.

The configurations (12_3) have their own somewhat convoluted history. The first enumeration of the combinatorial configurations (12_3) was carried out by Daublebsky [**D2**] in 1895, using the method of remainder figures. He found that only 18 different remainder figures could possibly occur in such a configuration. Through various arguments (described only in general terms) Daublebsky arrived at the conclusion that these remainder figures could be combined to yield many hundreds of configurations (12_3). Then he "... drew a schematic diagram of each configuration on a separate piece of paper ..." and determined for each the "remainder system", that is, a list of the different remainder figures occurring in the configuration. Finally, configurations with the same remainder system were investigated to see whether they are isomorphic. This turned out to be the case in most — but not all — cases. Daublebsky presented the resulting 228 combinatorial configurations by their configurations tables (these take 23 pages!!!). He also gave some other data and provided drawings for geometric realizations of a few of the configurations. In a later paper [**D3**], Daublebsky gave results of his investigations of the groups of automorphisms of each of the 228 combinatorial configurations (12_3). The first independent enumeration of the combinatorial (12_3) configurations was carried out only in 1990, by Gropp [**G2**]. It showed that Daublebski missed one, so that there are in fact 229 such configurations. (I assume that Gropp compared his list with that of Daublebsky, and that the one additional configuration is the only discrepancy between the two lists. A statement in [**G1**] can be interpreted this way.) Gropp communicated to me the configuration table of this configuration, and it can be read off from the illustration in [**D4**] and [**G6**]. As with configurations (11_3), the 229 combinatorial configurations (12_3) have been independently enumerated (by two different algorithms) in [**B4**].

The only published proof that all 228 combinatorial configurations (12_3) found by Daublebsky are geometrically realizable was given only recently, by Sturmfels and White [**S12, S13**]. Sturmfels and White also proved that all these (12_3) configurations are realizable in the rational plane. In a private communication, B. Sturmfels showed that the "new" combinatorial configuration found by Gropp is also geometrically realizable, even in the rational plane; a diagram is shown in Dorwart and Grünbaum [**D4**].

The numbers of different combinatorial configurations (n_3) have been determined for $n \leq 14$ by Gropp [**G2**], and for $n \leq 18$ by [**B4**]. See Table 1, which includes also the number for $n = 19$ computed by the same method. There seems to be no estimate of the asymptotic growth of the number of types of combinatorial configurations (n_3) as n goes to infinity.

In contrast to the above discussion, for $n \geq 13$ there is no general information available concerning the possible realizations or representations of the configurations (n_3) as geometric configurations of points and lines in the Euclidean plane. The only known connected configuration (n_3) with $n \geq 10$ which cannot be so represented is one of Kantor's (10_3) configurations, discussed above and indicated by the incorrect drawing in Figure 4. This observation and the results mentioned earlier lead to the conjectures:

CONJECTURE 3.1. *Every connected combinatorial configuration (n_3) with $n \geq$ 11 can be represented by points and lines in the real Euclidean plane.*

CONJECTURE 3.2. *Every configuration (n_3) that can be represented or realized by points and lines in the real plane can also be represented or realized in the rational plane.*

It should be stressed that the distinction between representability and realizability of a configuration (n_3) is very important. In Figure 1 we indicated by a

TABLE 1. The number of non-isomorphic combinatorial configurations (n_3), from [**B4**] and [**B7**].

n	Number of all (n_3) configurations	Selfdual (n_3) configurations
7	1	1
8	1	1
9	3	3
10	10	10
11	31	25
12	229	95
13	2,036	365
14	21,399	1,432
15	245,342	5,799
16	3,004,881	24,092
17	38,904,499	102,413
18	530,452,205	445,363
19	7,640,941,062	1,991,320

diagram a *representation* of a combinatorial configuration (16_3). This configuration (and many others) cannot be *realized* since, by the theorem of Pappus, the line L must pass through the point P with which it is not incident. However, all known examples of configurations (n_3) that are representable but not realizable are *2-connected* at most. Hence the following

CONJECTURE 3.3. *Every 3-connected configuration (n_3) that is representable in the real plane is realizable in the plane as well.*

As a sidelight to these conjectures it is appropriate to mention that the distinction between representation and realization of a configuration was very slow to be noted. For example, Schroeter [**S4**] describes very carefully the construction of the nine realizable configurations (10_3). He starts each construction by choosing a certain number of arbitrary points, and adding in some cases additional points on already constructed lines — but without noticing that some choices lead to representations that are not realizations. This is illustrated in Figures 5 and 6. Analogously, Steinitz in his fundamental theorem, which we shall discuss soon, claims to establish realizations of the configurations, or of near-configurations in which one incidence is not satisfied, — although this result is invalid. It becomes valid if representations are considered instead of realizations. However, there is a basic difference between the shortcomings of the Schroeter constructions and the Steinitz claim: The former can be made correct if assuming that the construction start with *generic* choices of points and lines, while no such correction can salvage Steinitz's assertion.

An error first committed by Schönflies [**S1**] is the claim that all combinatorial configurations (n_3) that are vertex-transitive and contain triangles are selfdual ("sich selbst reciprok"). In [**S2**] Schönflies states that this is a consequence of the (true) fact that if a configuration contains triplets of points that form triangles, then it must contain triplets of lines that form triangles — but this deduction is invalid; see Steinitz [**S9**, p. 307–309]. A related error is the frequently made *definition* by which configurations with the same number of points and lines are called

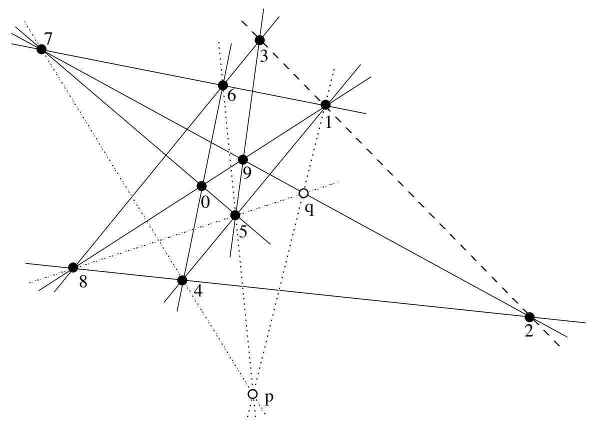


FIGURE 6. The same construction as in Figure 5 yields only a representation, since the point 1 lies on line 8-0-9.

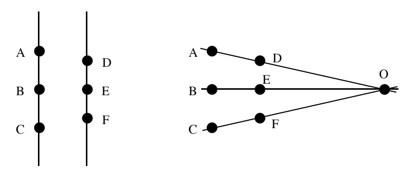


FIGURE 7. The Martinetti transformation.

"selfdual". As visible from Table 1, this happens to be true for configurations (n_3) with $n \leq 10$. However, already for $n = 11$ only 25 of the 31 configurations are selfdual; the fraction of self-dual configurations decreases rapidly with growing n — for $n = 16$ they form less than 1% of all configurations. This error of confusion was, unfortunately, committed also by Coxeter in [**C3**], and occurs still in some publications.

Another type of misleading terminology is still in wide use: Many authors (too many to list) call "*symmetric*" all configurations (or more general incidence structures) with the same number of points as lines. This is clearly inappropriate, since most of these objects admit no automorphisms or any other kind of incidence-preserving "symmetries". It would seem that "*balanced*" would be a far better designation — but if the use of this term is opposed because it occurs in the context of block designs, "*equinumerous*" might work as it does convey the meaning without implying properties that do not exist, and without impinging on other topics. With very few exceptions, in this paper we shall consider only equinumerous configurations.

3.2. A different direction in the studies of configurations (n_3) was initiated by Martinetti [**M1**]. Speaking of combinatorial configurations but using geometric language, he describes a simple operation which may be applied to a configuration (n_3) to obtain a configuration $((n+1)_3)$. The operation is schematically indicated in Figure 7. It replaces two "parallel" lines (that is, lines with no common points) such that corresponding pairs of points (AD, BE, CF) are not contained in any line

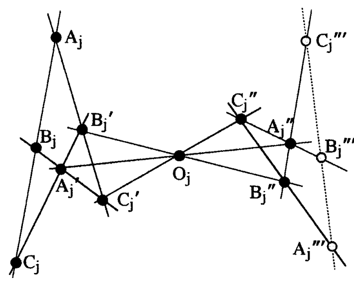


FIGURE 8. The "module" used in the Martinetti construction. Only the ten solid dots and the ten solid lines form one module.

of the configuration, by three lines that pass through these three pairs and a new point O. If a configuration $\mathcal{C}$ can be obtained from a smaller one by this operation, Martinetti calls $\mathcal{C}$ *reducible*; otherwise it is *irreducible*. Martinetti's main result is the claim that for each n there are very few irreducible (n_3) configurations, and he gives a complete description of all irreducible configurations. More precisely:

Theorem claimed by Martinetti (1887) [**M1**] *A connected* (n_3) *combinatorial configuration is irreducible if and only if it is one of the following:*

(i) *For* $n \geq 7$*, the cyclic configuration* $\mathcal{C}_3(n)$ *with lines* $[j, j+1, j+3]$ *(mod* n*), for* $0 \leq j \leq n-1$*;*
(ii) $n = 10m$ *for some* $m \geq 1$*, and the configuration is the one described below and denoted* $\mathcal{M}(m)$*;* $\mathcal{M}(1)$ *is the Desargues configuration* $(10_3)_1$*.*
(iii) $n = 9$*, and the configuration is the Pappus configuration* $(9_3)_1$*.*
(iv) $n = 10$*, and the configuration is* $(10_3)_2$ *or* $(10_3)_6$ *in the list of* (10_3) *configurations of Schroeter* [**S4**].

Martinetti's combinatorial configuration $\mathcal{M}(m)$ can best be explained as consisting of m copies of the family of the ten points indicated by solid dots in Figure 8, and the ten solid lines shown there. The jth copy is joined to the $(j+1)$st by identifying A_j''' , B_j''', C_j''' with A_{j+1}, B_{j+1}, C_{j+1}, respectively; all subscripts taken (mod m).

Martinetti's proof is, not surprisingly, involved and long. The result was quoted or mentioned many times over the next century; see, for example, Steinitz [**S8**, pp. 486–487], [**S10**, pp. 153–154], Gropp [**G1, G2, G6, G8**], Carstens et al. [**C1**]. In lecture notes for my configurations courses in 1999 and 2002 I wrote about the proof of Martinetti's theorem the following:

> I have not checked the details, and I do not know it as a fact that anybody has. The statement has been accepted as true for these 115 years, and it may well be true. On the other hand, Daublebski's enumeration of the (12_3) configurations was also considered true for a comparable length of time ...

As it turned out, my suspicion has been vindicated by the Ph.D. thesis of M. Boben [**B5**]; see also [**B6**]. He showed that Martinetti's list of irreducible configurations is incomplete. The error in Martinetti's proof arises as follows. When constructing $\mathcal{M}(m)$, he attaches m copies of the "module" in Figure 8 as indicated above; the mth copy is attached to the first "straight", by identifying A_n''' with

A_1, and similarly for the B's and C's, thus obtaining $\mathcal{M}(m)$. However, as shown by Boben, that attachment can also be done in "twisted" ways, two of which yield irreducible configurations which we may denote by $\mathcal{M}^*(m)$ and $\mathcal{M}^{**}(m)$. These are obtained by identifying A_n''' with C_1, B_n''' with B_1, and C_n''' with A_1 for the former, and A_n''' with C_1, B_n''' with A_1, and C_n''' with B_1 for the latter. A separate argument shows that the three resulting configurations are non-isomorphic for every m. With this modification, Boben's corrected version of Martinetti's result replaces parts (ii) and (iv) by the following:

(ii) $n = 10m$ *for some* $m \geq 1$, *and the configuration is one of* $\mathcal{M}(m)$, $\mathcal{M}^*(m)$ *or* $M^{**}(m)$ *described above. For* $m = 1$ *these are the configurations* $(10_3)_1$, $(10_3)_2$ *and* $(10_3)_6$.

3.3. A very interesting situation concerns Steinitz's Ph.D. thesis [**S6**]. In it, Steinitz proves a remarkable result, in an inspired way. However, he does not prove what he believes and asserts to have proved, but a considerably weaker result. It is hard to understand how Steinitz — a profound and very painstaking researcher — could make such a logical error; it is even harder to understand that the error was not detected for more than a century, despite the frequent mentions of Steinitz's theorem in the literature. The error came to light only in the presentation of T. Pisanski at the Ein Gev conference in 2000 [**P2**].

THEOREM 3.4 (Steinitz [**S6**] (in modern and correct formulation)). *For every connected combinatorial configuration* (n_3) *and every choice of one line (or of one point), there is a selection of distinct points and lines in the plane which represent all the incidences of the configuration except possibly the incidences of the chosen line (or the chosen point).*

Steinitz claimed to have proved the above assertion with "realize" instead of "represent". As stated explicitly in [**S9**], [**S8**, p. 485], or [**S10**, p. 150], Steinitz considers as geometric configurations only those that contain no incidences besides the ones of the combinatorial configuration. The failure of the realization claim follows at once from considerations of point F_1 (or line M) in Figure 1; the point B_2 has to be on line L with which it is not incident regardless of the incidences or nonincidences of F_1 (or M).

I feel humbled to realize that although I found the configuration shown in Figure 1 long ago (see [**D4**]), and although I lectured on Steinitz's theorem several times, I did not detect his error.

Steinitz's proof is remarkable enough to deserve a brief description. It has a combinatorial part, and a geometric one.

The centerpiece of the combinatorial part is the claim:

THEOREM 3.5 (Steinitz [**S6**]). *Every combinatorial configuration* (n_k) *admits an orderly configuration table.*

Here a configuration table for an (n_k) configuration is said to be *orderly* if every *row* of the table contains all the points (hence each precisely once).

A statement that Theorem 3.5 holds for $k = 3$ (without any justification or hint of proof) appears in Martinetti [**M1**]. Most later authors do not mention the result — much less its proof — although many writers seem to accept it as self-evident. On the other hand, the statement in Page and Dorwart [**P1**] regarding this result is incorrect, as are the consequences deduced by them from the erroneous statement.

As pointed out by Gropp [**G7**], Theorem 3.5 implies the later result known as König's theorem [**K5**], that every bipartite graph of constant valence contains a factor of degree 1.

Having established that any given (n_3) configuration has an orderly configuration table, after a few additional steps, Steinitz arrives at the result that, having chosen a point or a line, it is possible to arrange all elements (that is, points and lines) of the combinatorial configuration in such a sequence that each is incident with at most two elements preceding it in the sequence, the only exception being the chosen element, which is the last in the listing. Now, in the geometric part, since an element incident with at most two previous ones can obviously be constructed, it follows that the combinatorial configuration can be represented by points and straight lines that satisfy all the incidences, except those of the last line. Clearly, if the last three points are not incident with a (straight) line, they can always be made incident with a curve of degree 2. This is the result of Theorem 3.4.

Some additional comments seem appropriate.

First, from the proof it is obvious that the whole construction could be carried out in the rational plane, so that all the points and lines (including the last one, if it exists), are rational.

Second, Steinitz devotes more than half the dissertation [**S6**] (24 pages) to a consideration of ways in which one could guarantee that the final step in the above proof can be made using a straight line instead of a curve of degree 2. While this might be another interesting result, I have not been able to follow the exposition. (In fact, I know of nobody who claims to have understood and verified this part of [**S6**].) The opaqueness of the exposition can best be seen from the last two sentences of Steinitz's introduction to this part of the work (see [**S6**, p. 22]):

> ... Without any particular assumptions about the configurations, a method will be presented below following which one can reach a linear presentation. However, for each configuration to which we want to apply this method, an additional investigation is necessary since the method becomes illusory in certain cases. [My translation]

In mentioning [**S6**] in the survey [**S8**, p. 490], Steinitz is equally uninformative. Stating that his method is an exte nsion of Schroeter's approach in [**S3, S4**], he ends the explanation by stating:

> Schroeter's method can be generalized so that it is applicable to most configurations (n_3).

It seems that the "method of Schroeter" is rooted in arguments due to Möbius in the early part of the nineteenth century, in particular in [**M3**].

Third, even if the proof is valid, and somebody were to make the exposition understandable — this would prove our Conjecture 3.1, but it would not be a proof of the analogue of Conjecture 3.1 for *realizations*, as claimed by Steinitz. Indeed, we know from examples such as the one in Figure 1 that some representable configurations are not realizable, hence Conjecture 3.1 cannot be generally valid for realization.

3.4. Interpretations of configurations as polygons or families of polygons go back to the very beginning of the study of configurations (n_3). The connection arises by considering segments of the lines of a configuration determined by the

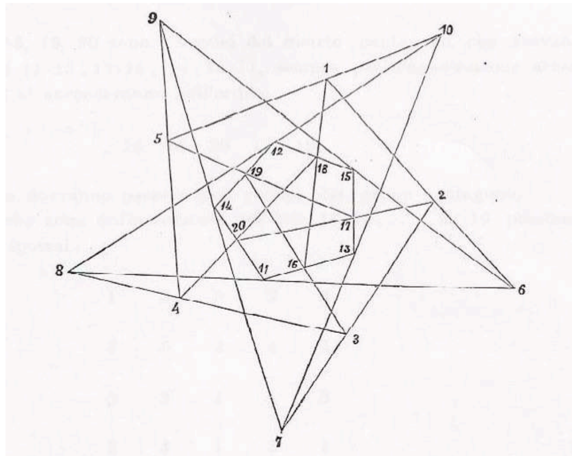


FIGURE 9. The seemingly first published graphical presentation of a polycyclic configuration, specifically a 4-cyclic (20_3) configuration. (From Visconti [**V1**].) There are four pentagons: $1,2,3,4,5$; $6,8,10,7,9$; $11,14,12,15,13$, and $16,19,17,20,18$. Each is inscribed in the preceding one, and the first one in the last.

points of that line; such segments can be used to form one or more circuits — polygons. Utilizing the old and ever-present confusion between lines and segments, it is customary to say that one polygon (in the configuration) is *inscribed* into another polygon if the vertices of the former are configuration points (other than the endpoints) of the lines determined by the sides of the latter. The latter is also said to be circumscribed about the former. To clarify this description, consider Figure 9. As specified in the caption, the lines there form four pentagons, each inscribed into another and circumscribed about a third.

Such families of mutually inscribed/circumscribed polygons have been discussed very frequently — for example, by Kantor [**K3, K4**], Martinetti [**M1**], Schöenflies [**S1, S2**], Steinitz [**S9**], and many others. In some cases the discussions concern combinatorial configurations, in other configurations in complex or other planes. Mostly it is assumed that the inscription is "regular", by which is understood that the order of sides of a polygon and the order of the vertices of the inscribed polygon coincide. The concepts have been used to generate various families of configurations. However, several of the assertions found in these papers are not true. As they are of no particular relevance for our discussion, we shall not give details. In Sections 4, 5, and 6 we shall discuss developments that can be interpreted as streamlined families of such polygons.

One special case deserves particular mention: In many cases an (n_3) configuration can be presented as an n-gon inscribed and circumscribed to itself. Interpreting this situation via the Levi graph it is clear that such an n-gon represents a *Hamiltonian circuit* in the configuration. The circuit passes through each point precisely once, and utilizes each line precisely once. As mentioned earlier, Kantor [K4] asserted that every configuration (n_3) has a Hamiltonian circuit. This was disproved by Steinitz [**S7**] by an example with $n = 28$. The smallest example of this kind is shown in Figure 10 (from [**D4**]). Steinitz [S7] also mentions that he verified

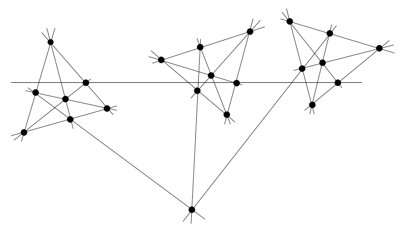


FIGURE 10. The smallest known non-Hamiltonian configuration (n_3); here n=22.

the existence of Hamiltonian circuits for $n \leq 11$. Nothing seems to be known for $12 \leq n \leq 21$.

CONJECTURE 3.6. *All connected combinatorial (n_3) configurations with $n \leq 21$ are Hamiltonian.*

Both the $n = 28$ example of Steinitz, and $n = 22$ example in Figure 10 are only 2-connected. Until recently, I believed that 3-connected configurations (n_3) always have a Hamiltonian circuit. However, this is not the case: There is a 3-connected geometric configuration (50_3) with no Hamiltonian circuits (see [**G13**]).

CONJECTURE 3.7. *Every connected (n_4) configuration has a Hamiltonian circuit.*

4. Configurations (n_4)

The history of configurations (n_4) is much shorter than that of (n_3). It seems that among the first to publish examples of such configurations, both combinatorial and geometric, while aware of the need to distinguish between them, was Brunel [**B11**] in 1898; this work seems to have escaped attention of all later writers. In an earlier paper Brunel followed the ideas of a polygon inscribed and circumscribed to itself, which had been quite popular at the time, as a special class of (combinatorial or geometric) configurations (n_3). In [**B11**] he pursued this idea farther, by considering a polygon *doubly* "inscribed and circumscribed" to itself. In both situations we would call such polygons "Hamiltonian circuits" in the configuration. Each line of such a polygon is incident, besides the two points (vertices of the polygon) that define it as a side of the polygon, with precisely two additional vertices of the polygon. He determines that any combinatorial configuration (n_4) must satisfy $n \geq 13$.

In [**B11**] Brunel gives two constructions. In the first, he presents an orderly configuration table, and states that while the verification that this indeed determines a combinatorial configuration (35_4) is easy, the graphical representation requires some effort. From this (especially in view of his later comments) one may conclude that he had a geometric realization of this configuration. In fact, this configuration turns out to be isomorphic to the geometric configuration (35_4) mentioned in [**G14**], communicated to the authors by Ludwig Danzer. Although no reasonable diagram of this configuration seems to be available, it can be described easily enough by a construction of the kind used by Cayley and others in similar context a century and a half ago. In the case under discussion, start with seven points in general position in real 4-space; consider the 35 2-planes and 35 3-spaces they generate, and intersect this family by a 2-dimensional plane in general position to obtain the required geometric configuration (35_4).

Brunel's second construction yields combinatorial configurations (n_4) on which a cyclic group operates transitively. This includes the explicitly specified configurations for $13 \leq n \leq 16$, but the results presented are marred both by typos, and by outright errors. Without noticing their abstract isomorphism, in several cases Brunel lists isomorphic doubly selfinscribed and selfcircumscribed polygons as distinct. For example, in case $n = 13$ Brunel lists translates of $\{0,1,4,6\}$ and $\{0,1,3,9\}$ as the two polygons, although the permutation

$$(0)(1)(2)(3,4)(5)(6,9,8,10,12,7)(11)$$

maps the first polygon onto the second. But even allowing for these shortcomings, we see that Brunel anticipated the corresponding results of Merlin [**M2**], and even went a bit beyond them. A corrected list would show one cyclic configuration (or polygon) for $n = 13$ and 14, three for $n = 15$, and two for $n = 16$. This coincides with the recent list of cyclic configurations given by Betten and Betten [**B3**], to which we shall return soon. Brunel also noted that translates of $\{0,1,4,6\}$ yield a configuration for all $n \geq 13$; this anticipated a result of Gropp [**G2**].

Merlin mentions in [**M2**] that configurations (n_4) have not been investigated systematically, although some isolated ones were discovered by F. Klein [**K1**], W. Burnside [**B12**], and others. He constructs a combinatorial configuration (13_4) and proves its uniqueness and minimality. He also constructs a configuration (14_4) and proves it is unique. Merlin states that there are exactly *three* distinct configurations (15_4) which, however, are not presented. In fact, he is mistaken. As shown by Betten and Betten [**B3**], there are *four* different configurations (15_4), three of which are cyclic and coincide with the three doubly selfinscribed and selfcircumscribed polygons of Brunel (who did not comment on the possibility of noncyclic configurations (15_4)). In the same context, Merlin makes two additional errors: (i) He claims that his three configurations (15_4) can be distinguished by the number of vertex-disjoint triangles present in them, which he claims to be 5, 1 and 0, respectively. In fact, all four configurations (15_4) have five such triangles, the maximal possible number. (ii) He states that his configurations (13_4), (14_4), and (15_4) have orderly configuration tables (which is correct and proved by Steinitz in [**S6**] for all configurations (n_k)), and states that it follows that there is no Hamiltonian circuit for any of them — which is wrong. Steinitz's orderliness result has no such implications, and Brunel's explicit constructions in [**B11**], of which Merlin is unaware, provide counterexamples to Merlin's claim.

By a construction analogous to the one devised by Martinetti for configurations (n_3), Merlin shows that for every $n \geq 30$ there are combinatorial configurations (n_4). In fact, it is easy to show that there are such configurations for all $n \geq 13$. As to the number $N(n)$ of distinct combinatorial configurations (n_4), the only known values are those given by Betten and Betten [**B3**], namely $N(13) = N(14) = 1$, $N(15) = 4$, $N(16) = 19$, $N(17) = 1972$, and $N(18) = 971171$. For $n \geq 15$ these numbers have not been independently verified.

As far as is known, none of these combinatorial configurations is geometrically realizable. Merlin [**M2**] shows that the configurations (13_4), (14_4), and (15_4) are not geometrically realizable. But he also notes that geometric configurations (n_4) do exist for infinitely many values of n. His construction uses "stacks" of configurations of type $[3,3]$ and vertical lines through their vertices to construct configurations of

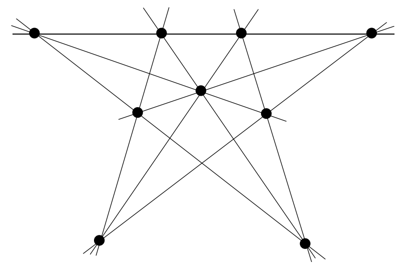


FIGURE 11. A set of nine points and nine lines that is realizable in the real Euclidean plane but is not realizable in the rational Euclidean or projective plane.

type [4, 3], and then stacks of duals of the projections of these into the plane to construct configurations of type [4, 4]. While this yields geometric configurations (n_4) for infinitely many values of n, there are infinitely many n that are not covered.

The latest result in this direction, established by a variety of mostly *ad hoc* constructions, is given in [**G12**], the last of several papers on this topic:

THEOREM 4.1. *Connected geometric configurations* (n_4) *exist for all* $n \geq 21$, *except possibly for the following ten values of* n*:* $22, 23, 26, 29, 31, 32, 34, 37, 38, 43$.

CONJECTURE 4.2. [**G12**] *No combinatorial configuration* (n_4) *with* $n \leq 20$ *is realizable.*

It is also highly probable that there exist no geometric configurations (n_4) for the ten values of n listed in Theorem 4.1.

CONJECTURE 4.3. *No combinatorial* (n_4) *configuration on which a cyclic group acts transitively has geometric realizations.*

It should be noted that in contrast to Conjecture 3.2, there exist (n_4) configurations that can be realized in the Euclidean plane but not in the rational plane. The simplest construction I know starts with the collection of nine points and nine lines shown in Figure 11. It is well known (see [**G9**, Section 5.5], [**B9**, pp. 5, 40]) that this “partial configuration” cannot be represented in the rational projective (or Euclidean) plane, but it is easily seen that it can be imbedded into an (n_4) configuration with $n \leq 44$.

5. Highly symmetric configurations: Astral configurations

A considerable part of the recent interest in geometric configurations is due to the results obtained in the study of configurations that have a large degree of geometric symmetry.

Configurations (n_3) and (n_4) must have at least two orbits of points and two orbits of lines; by the terminology introduced in Section 2, configurations with these minimal numbers of orbits are called astral. Some instances of astral (n_3) configurations have appeared earlier, but without any systematic considerations; we shall return to the historical examples and references is Section 6.

In general, there are three classes of h-astral configurations (n_k):

(i) Configurations h-astral in the extended Euclidean plane E^{2+} but not contained in the Euclidean plane E^2 itself.

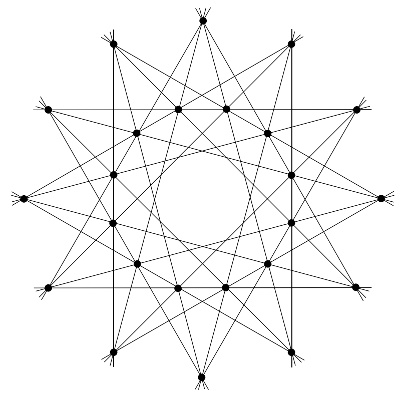


FIGURE 12. The smallest astral configuration of type [4,4]: the configuration (24_4), which has symbol $12\#(5,4,1,4)$.

(ii) Configurations h-astral in E^2, with a cyclic group of symmetries; we call them *h-cyclic*, or *polycyclic* if the value of h is not relevant.
(iii) Configurations h-astral in E^2, with a dihedral group of symmetries; we call them *h-dihedral*, or *polydihedral* if the value of h is not relevant.

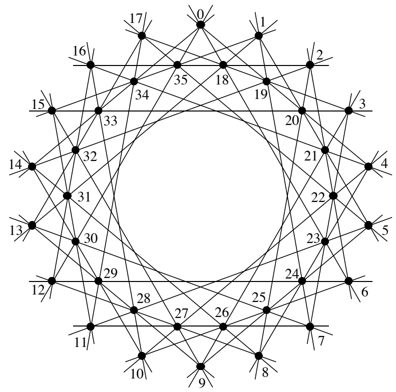
It should be noted that for even k, all h-astral configurations (n_k) are h-dihedral. For $k = 3$ there exist astral (that is 2-astral) configurations of all three classes. We shall discuss additional results later.

The remaining part of this section is devoted to h-astral configurations (n_4), and in particular the astral ones among them. These configurations are the best explored kind, hence we discuss them first. Since they are all polydihedral, we shall simplify the terminology and call them *h-astral* resp. *astral* for short.

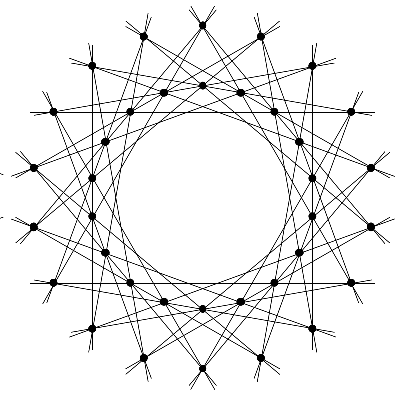
The first drawing of an astral (n_4) configuration (or of any (n_4) configuration!) seems to have appeared in [**G14**]; it is the (24_4) shown in Figure 12. This was the beginning of a development I would like to sketch now in a few words.

I had been wondering what other (n_4) configurations exist for which there are just two transitivity classes of points (and two classes of lines) under isometric symmetries of the configuration — that is, astral configurations. Manual drafting of diagrams soon reached the limits of reliability. At about that time, Stan Wagon gave a series of talks at the University of Washington, extolling the virtues and ease of use of the *Mathematica* software as both a computational tool and a graphic one. As it turned out, it was really easy to write a program that tested for which *spans* of diagonals of an m-gon do appropriate intersection points of the diagonals, together with the vertices of the m-gon, form a configuration (n_4), where $n = 2m$. (The *span* of a diagonal in a convex polygon is the number of edges spanned by the diagonal.) The first of the experimental results showed that this happens if and only if $n = 2m = 12k$ for some integer $k \geq 2$. Figure 12 shows the case $k = 2$, while Figure 13 shows six astral configurations with $k = 3$ (that is, (36_4) configurations).

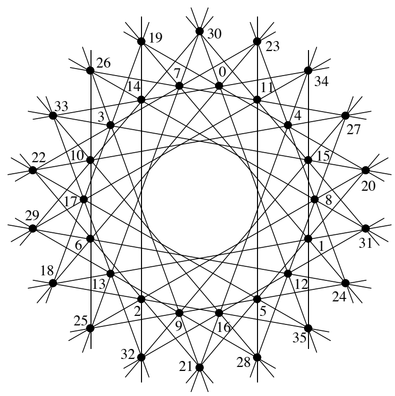
The more detailed study of which pairs of spans of diagonals of a polygon with $m = 6k$ sides yield 2-astral configurations led to a puzzling situation, which is illustrated in Figure 14. (In order not to prejudice between the two spans s and t, both points (s,t) and (t,s) are indicated; the hollow circles denote pairs for which there are two different configurations for the same spans.) The valid pairs seemed



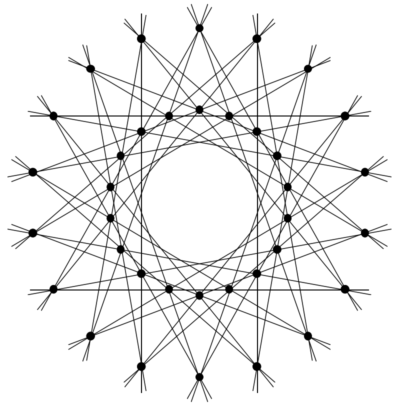
18#(6, 5, 1, 4)



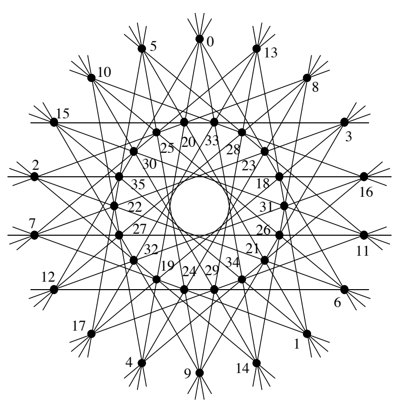
18#(6, 4, 1, 5)



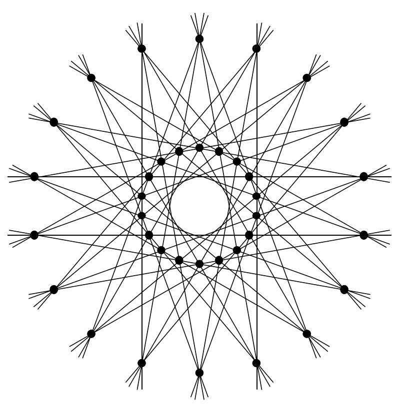
18#(7, 6, 2, 5)



18#(7, 5, 2, 6)



18#(8, 7, 1, 6)



18#(8, 6, 1, 7)

FIGURE 13. The six different astral configurations (36_4), with their symbols. The two configurations in each row are polars of each other. The labeling of the three configurations at left indicates an isomorphism between them.

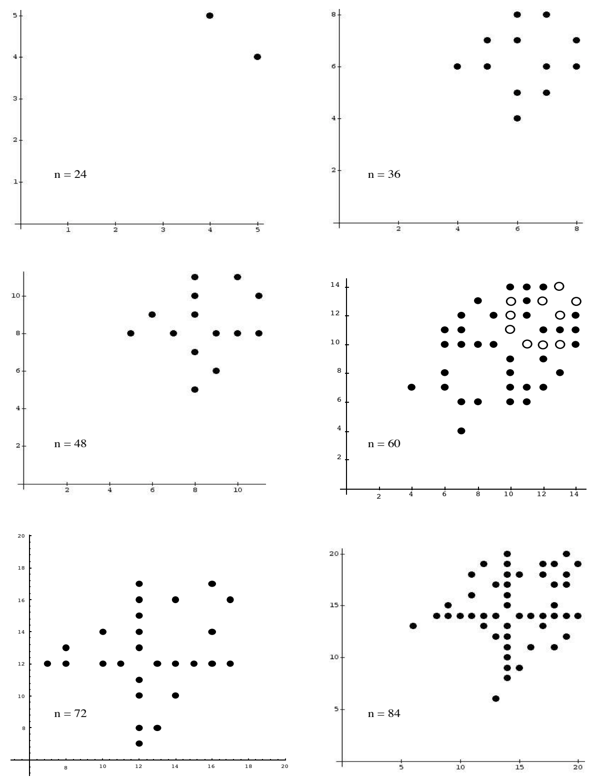


FIGURE 14. Plots of the parameters of existing astral configurations (n_4).

to exhibit no visible regularity. But as the experimentation continued to larger values of k, illustrated in Figure 15, a clear indication emerged: Most of the pairs satisfy simple linear relations, with only a few “sporadi c” pairs besides. This led to the trigonometric verification that the experimentally obtained pairs do in fact correspond to astral configurations, and to the listing of such configurations known to exist given in Theorem 5.1 (see [**G11**], where a slightly different notation was used; two errors in the data of [**G11**] have been corrected in [**B1**]).

THEOREM 5.1. *There exist two infinite families of astral configurations* (n_4), *that can be described by the symbols* $(6k)\#(3k-j, 2k, j, 3k-2j)$ *where* $k \geq 2$ *and* $1 \leq j \leq k-1$, *and* $(6k)\#(2k, j, 3k-2j, 3k-j)$ *where* $k \geq 2$, $1 \leq j \leq 2k-1$ *but* $j \neq k$ *and, if* k *is even,* $j \neq 3k/2$. *There also exist 27 basic sporadic configurations listed in Table 2, and their multiples.*

The conjecture that the astral configurations specified in Theorem 5.1 are the only ones was expressed in [**G11**], and established by Leah Berman in her University of Washington doctoral thesis in 2002. This can be formulated as Theorem 5.2, which was published in [**B1**].

THEOREM 5.2 (Berman). *The following is a complete list of astral configurations* (n_4)*:*

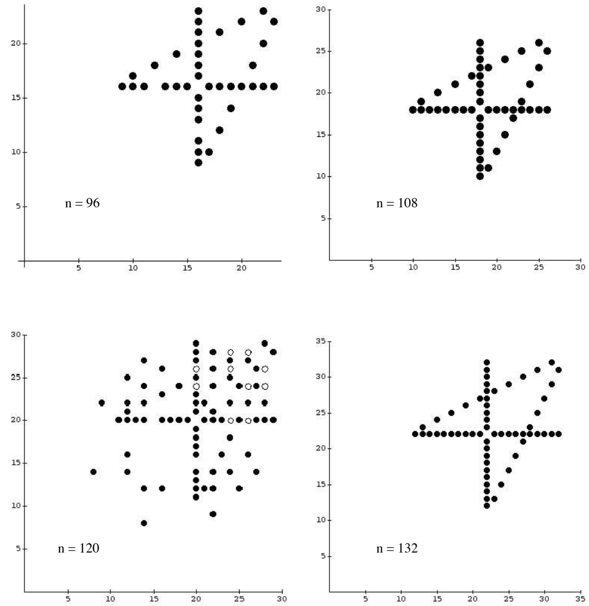


FIGURE 15. More plots of the parameters of existing astral configurations (n_4).

TABLE 2. Symbols of the 27 basic sporadic astral configurations (n_4). The full meaning of the symbols $m\#(s_1, t_1, s_2, t_2)$ will be explained below. Here it suffices to note that $n = 2m$, and that s_1 and t_2 are the spans of the diagonals of the m-gon.

30#(7,6,1,4)	30#(7,4,1,6)	30#(8,6,2,6)	30#(11,10,1,6)
30#(11,6,1,10)	30#(12,10,6,10)	30#(12,11,2,7)	30#(12,7,2,11)
30#(13,12,1,8)	30#(13,8,1,12)	30#(13,12,7,10)	30#(13,10,7,12)
30#(14,13,6,11)	30#(14,11,6,13)	30#(14,12,4,12)	42#(13,12,1,6)
42#(13,6,1,12)	42#(18,17,6,11)	42(18,11,6,17)	42#(19,18,5,12)
42#(19,12,5,18)	60#(22,21,2,9)	60#(22,9,2,21)	60#(25,24,5,12)
60#(25,12,5,24)	60#(27,26,3,14)	60#(27,14,3,26)	

(i) *Configurations having vertices on two concentric regular m-gons, where $m = n/2$, that are listed in Theorem 5.1;*

(ii) *Configurations having vertices not on two concentric regular m-gons that result by taking two concentric copies of one of the configurations in part (i), rotated with respect to each other through any angle other than a multiple of π/m.*

Berman's proof relies in a non-trivial way on the characterization of multiple intersection points of diagonals in regular polygons. Although that question appears to be a simple problem it is, in fact, surprisingly deep and delicate. It was first solved (except for minor glitches) by Bol [**B10**] but his result was apparently forgotten until the work of [**R3**], where the solution was presented in a different form. The

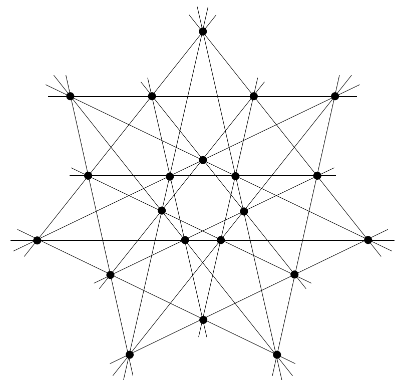


FIGURE 16. The 3-astral configuration (21_4), which is the smallest known geometric configuration (n_4). It can be described by the symbol $7\#(3,2,1,3,2,1)$, the derivation and meaning of which is described in the text.

solution was presented in a much more accessible and more general form by [**P3**] in 1998. (In the intervening years, various special cases have been rediscovered by several authors. Many references can be found in [**P3**].)

The enumeration of astral configurations (n_4) given in Theorems 5.1 and 5.2 dealt with *geometrically* distinct configurations. Very little is known about the *isomorphism* classes of these configurations. One rather unexpected result is that the six astral configurations (36_4) shown in Figure 13 are of only two non-isomorphic types. The configurations in the left column are isomorphic to each other by the labeling shown, while those on the right must therefore be isomorphic to each other, since they are polars of the ones at left. It is not hard to show that the configurations of each polar pair are not isomorphic. We venture:

CONJECTURE 5.3. *For each $n = 12k$, with $k \geq 3$, not all connected astral configurations (n_4) are isomorphic.*

This is probably easy to settle by a suitable generic example. The really hard problem in this context is finding how many non-isomorphic astral configurations (n_4) are there for each n.

I also offer

CONJECTURE 5.4. *No astral configuration (n_4) is isomorphic to a geometric configuration in the rational projective plane.*

The known results concerning h-astral configurations with $h \geq 3$ are far less complete. The first example — the unique 3-astral configuration (21_4) described in [**G14**] and shown in Figure 16 — is the smallest of all known *geometric* configurations (n_4), h-astral or not. The underlying *combinatorial* configuration has been known for a long time (first described, it seems, by Klein [**K1**] in 1879), and its realizations in the complex plane or in finite planes were investigated by many people (see Burnside [**B12**], Coxeter [**C6**], and the references in [**C6**]).

The study of h-astral configurations was greatly advanced by the work of Boben and Pisanski [**B8**], who called them *h-cyclic* or *polycyclic*. Their ideas were among those utilized in [**G12**] in the proof of Theorem 4.1. The following account differs

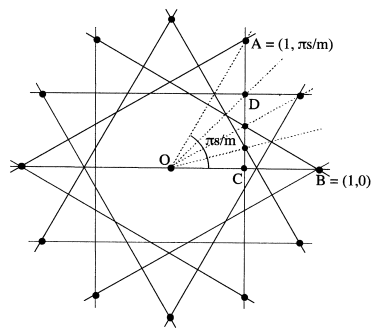


FIGURE 17. The notation for intersection points of diagonals of span s of a regular n-gon. Assuming the polygon to have unit circumcircle, the label $[[p, q]]$ for a point means that it is situatd on the diagonal of span s, and that it is the qth among the intersections of that diagonal with other diagonals of the same span, counting from the midpoint of the diagonal. For example, the point D has label [[4,3]].

in details from the approach in [**B8**]; it was developed during the presentation of the material in courses I gave at the University of Washington, and in preparation for a workshop in Bled (Slovenia) which unfortunately I could not attend.

The starting point for almost all h-astral configurations is a notation for *intersection points* of diagonals of span s of a regular m-gon, illustrated in Figure 17 and explained in its caption. (One exception is described below, and shown in Figure 23.) A notation for a connected h-astral configuration $\mathcal{C}$ can be devised using a *characteristic path* as follows:

Choose an orientation, say counterclockwise, and orient all diagonals of the regular m-gons (that is, lines of the configuration) accordingly. Choose an arbitrary point P_0 of $\mathcal{C}$ as starting point of the path, and through it an arbitrary line L_1 of $\mathcal{C}$ for which P_0 is the earlier of the two points in the same orbit. On L_1 we take the first point (according to the order on L_1) of the other orbit of points incident with L_1, and denote it P_1. Through P_1 pass lines of an orbit other than that of L_1; we choose as L_2 that one for which P_1 is the earlier point. Continuing in this way, after a finite number h of steps we return to the first orbit of points. The chosen points determine the characteristic path. Hence the polygons are m-gons, where $m = n/h$ and the configuration is h-astral. Then the configuration $\mathcal{C}$ can be described by a symbol of the form $m\#(s_1, t_1, s_2, t_2, s_3, t_3, \ldots, s_h, t_h)$, where s_j is the span of the diagonal L_j of the jth polygon, and $[[s_j, t_j]]$ is the symbol of the point P_j on L_j. It should be noted that, in general, the point P_h will not coincide with P_0. Note that necessarily each entry is a positive integer, different from the adjacent ones (understood cyclically), and is less than $m/2$.

The characteristic path $P_0P_1P_2 \ldots P_h$ describes and determines the configuration. It is indicated by a heavy gray line in the examples in Figures 18 and 19. Different starting points and choices of lines lead to equivalent symbols; they result by cyclic permutations of each other that advance through an even number of places, or else by a reversal of these. On the other hand, it is easy to show that advancing an odd number of places (and reversals of these) lead to symbols of the configuration polar to the starting one. Details can be found (with slightly different notation and terminology) in [**B8**].

The entries in a symbol $m\#(s_1,t_1,s_2,t_2,s_3,t_3,\ldots,s_h,t_h)$ cannot be chosen arbitrarily. It is not hard to verify that the symbol must satisfy the conditions

$$(*) \qquad s_1+t_1+s_2+t_2+s_3+t_3+\cdots+s_h+t_h \qquad \text{is even}$$

and

$$(**) \qquad \frac{\cos(\pi s_1/m)}{\cos(\pi t_1/m)}\,\frac{\cos(\pi s_2/m)}{\cos(\pi t_2/m)}\cdots\frac{\cos(\pi s_k/m)}{\cos(\pi t_k/m)}=1$$

If these necessary conditions are satisfied, a *combinatorial* configuration (n_4) can always be constructed, simply by following the indications of the path $P_0P_1P_2\ldots P_h$. However, to assure connectedness, besides conditions $(*)$ and $(**)$ we need:

$(***)$ If $m,s_1,t_1,s_2,t_2,s_3,t_3,\ldots,s_h,t_h$ have a common factor $f>1$, then the numbers m/f, s_1/f, t_1/f, s_2/f, t_2/f, s_3/f, t_3/f, …, s_h/f, t_h/f fail to satisfy at least one of the conditions $(*)$ and $(**)$.

The conditions $(*)$, $(**)$, and $(***)$ are sufficient to assure the existence of a *representation* of the combinatorial configuration implied by the symbol

$$m\#(s_1,t_1,s_2,t_2,s_3,t_3,\ldots,s_h,t_h).$$

However, according to [**B8**], for *realization* (rather than representation) an additional condition is required:

$(****)$ No proper subsequence $(s_i,t_i,s_{i+1},t_{i+1},\ldots,s_k)$ yields a symbol

$$m\#(s_i,t_i,s_{i+1},t_{i+1},\ldots,s_k,t^*)$$

that satisfies conditions $(*)$, $(**)$, and $(***)$, where $1\le t^*<m/2$. Dually, no proper subsequence $(t_i,s_{i+1},t_{i+1},\ldots,s_k,t_k)$ can yield a symbol

$$m\#(s^*,t_i,s_{i+1},t_{i+1},\ldots,s_k,t_k)$$

that satisfies conditions $(*)$, $(**)$, and $(***)$, where $1\le s^*<m/2$.

In Figure 20 we show an example of a symbol that fails condition $(****)$ and hence leads to a representation that is not a realization.

The h-astral configurations (n_4) can be classified into three types:

- *Trivial* are configurations for which the symbol is such that the (unordered) *set* of s_j's coincides with the set of t_j's. Hence conditions $(*)$ and $(**)$ are satisfied trivially, with no need to calculate the trigonometric functions. An example is the configuration (21_4) shown in Figure 16, with symbol $7\#(3,2,1,3,2,1)$.
- *Systematic* are configurations having symbols which are derived by explicit formulas for the parameters, for infinitely many values of n. Thus the satisfaction of condition $(**)$ is verifiable by manipulations of the trigonometric functions, again without the necessity to actually calculate the values of these functions.
- *Sporadic* are configurations that are neither trivial nor systematic.

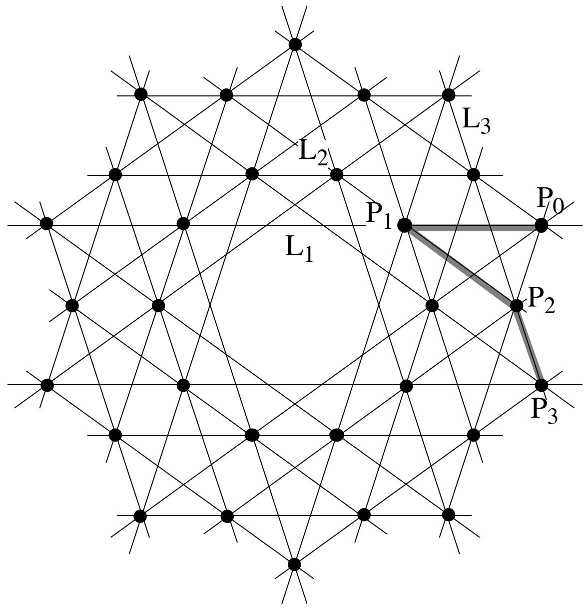


FIGURE 18. A 3-astral configuration (30_4), with symbol $10\#(4,3,1,3,1,2)$. This symbol is composed of the label [[4,3]] of point P_1, label [[1,3]] of point P_2, and label [[1,2]] of point P_3. The spans of the lines L_j are 4,1 and 1, respectively. Note that the parameter t_j in the label $[s_j, t_j]$ of the point P_j can be interpreted as the span of L_j in the polygon formed by the points in the orbit of P_j.

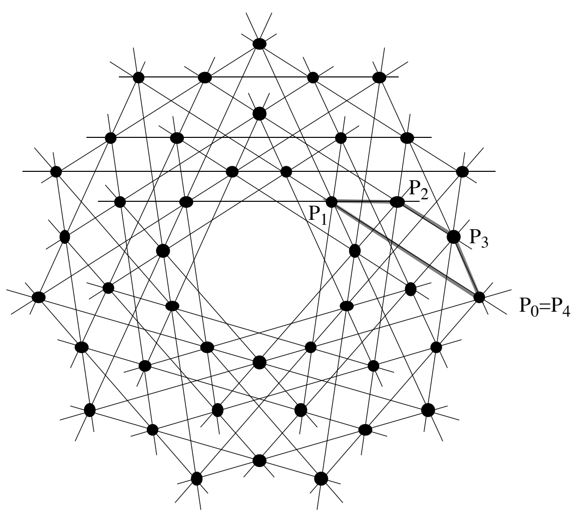
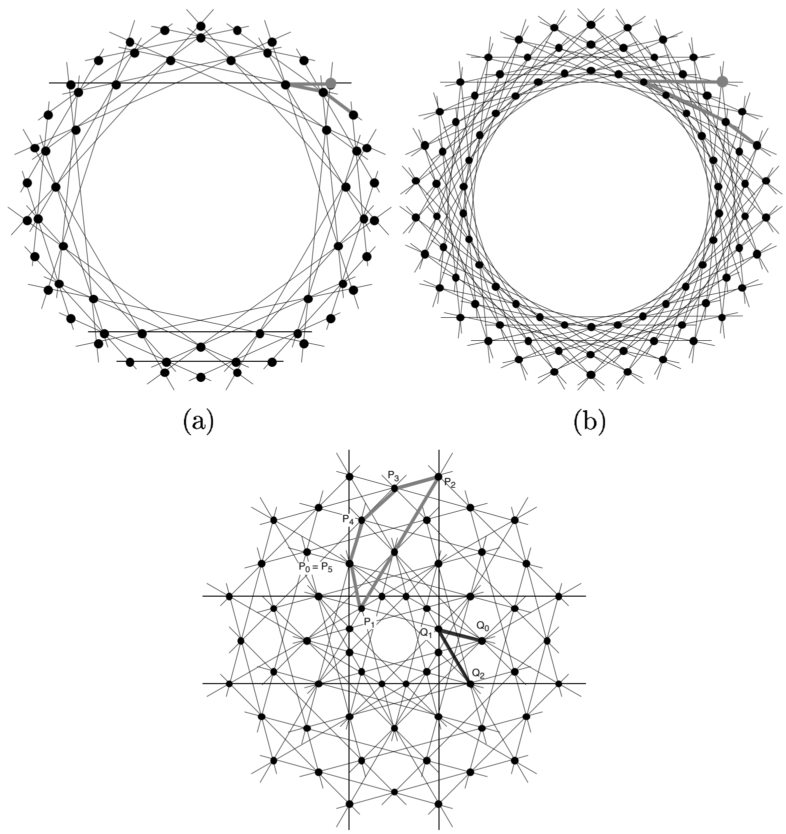


FIGURE 19. A 4-astral configuration (44_4), with symbol $11\#(4,1,3,4,2,3,1,2)$. Note that P_1 has label [[4,1]], P_2 has label [[3,4]], P_3 has label [[2,3]], and $P_4 = P_0$ has label [[1,2]].



(c)

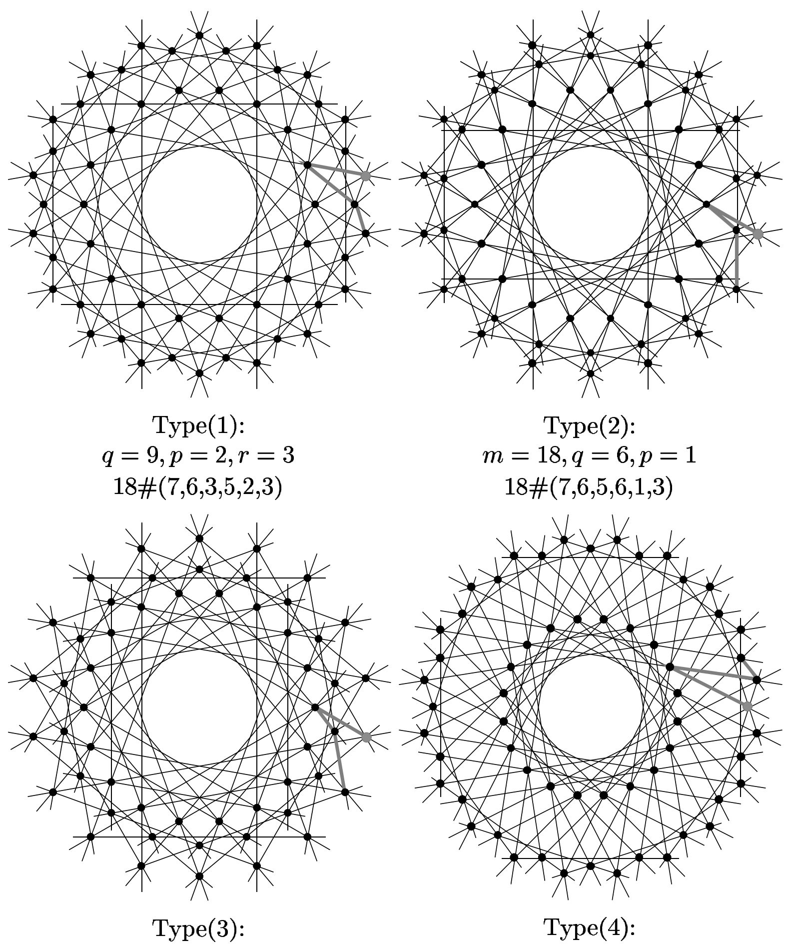
FIGURE 20. In each of the examples, a characteristic path starts at an enlarged dot and is shown in gray. (a) The necessity of condition $(*)$ is illustrated by the 3-astral symbol $15\#(4,3,2,3,1,2)$, which does not lead to an (n_4) configuration. (b) The entries of the symbol $30\#(8,4,2,6,4,6)$ have a common factor 2, but this symbol leads to a connected configuration since condition $(*{*}*)$ is satisfied. (c) The symbol $12\#(5,4,1,5,2,1,3,2,4,3)$ satisfies condition $(*)$, $(**)$ and $(* * *)$ but not $(* * **)$. This is not a realization of the symbol, since lines of one orbit are incident with six points each, and points of one orbit are incident with six lines each. The explanation is that if we take only the first four parts of the list in parentheses, and change the terminal 5 to a 4, we obtain a valid symbol $12\#(5,4,1,4)$ illustrated by the characteristic path $Q_0Q_1Q_2$. This is the symbol of the configuration shown in Figure 12, which can easily be seen as formed by the two innermost orbits of points and lines.

As visible from Theorems 5.1 and 5.2, astral configurations are either systematic or sporadic. However, for each $h \geq 3$, there are trivial h-astral configurations as well.

In unpublished work, L. Berman has identified four families of systematic 3-astral configurations $m\#(a, b, c, d, e, f)$:

$$\begin{aligned}
&(5.1)\quad m = 2q,\ \{a, c, e\} = \{q-p, q-2r, p\}, \quad \{b, d, f\} = \{q-2p, q-r, r\} \\
&(5.2)\quad m = 3q,\ \{a, c, e\} = \{q+p, q-p, p\}, \quad \{b, d, f\} = \{q, q, 3p\} \\
&(5.3)\quad m = 6q,\ \{a, c, e\} = \{3q-p, r, p\}, \quad \{b, d, f\} = \{3q-2p, 2q, r\} \\
&(5.4)\quad m = 10q,\ \{a, c, e\} = \{5q-p, 2p, p\}, \quad \{b, d, f\} = \{|5q-4p|, 4q, 2q\}.
\end{aligned}$$

Examples of configurations in each of these families are shown in Figure 21.



Type(1):
$q = 9, p = 2, r = 3$
18#(7,6,3,5,2,3)

Type(2):
$m = 18, q = 6, p = 1$
18#(7,6,5,6,1,3)

Type(3):
$m = 18, q = 3, p = 2, r = 4$
18#(7,6,4,5,2,4)

Type(4):
$m = 20, q = 2, p = 3$
20#(7,4,6,8,3,2)

FIGURE 21. Examples of systematic 3-astral configurations of the four known systematic types. One characteristic path is indicated in each, with the starting point shown by an enlarged gray dot.

In Figure 22 are shown several h-astral configuration (n_4) that exhibit various unusual phenomena. Some of these are easy to explain, but some are rather puzzling.

There are many open problems concerning h-astral configurations (n_4). A few of the more striking are:

- A complete characterization of representations of 3-astral configurations. This should take into account also configurations such as the one in Figure 23, found by L. Berman (private communication). This configuration falls outside the scope of description by symbols we used here. It is not known whether there are other configurations of this kind — it is hard to imagine that there is only a single one!
- What is the explanation for configurations such as the one in Figure 22(b), that are describable by our symbols but incompatible with condition $(****)$?
- Is it possible for an h'-astral configuration to be isomorphic to an h''-astral configuration with $h' \neq h''$?
- Are there finitely many basic sporadic h-astral configurations for each h?
- What are the possibilities of (n_4) configurations of the kind shown in Figure 24, which is (2, 3)-astral (in the notation introduced in Section 2)? What other unequal pairs of numbers of orbits can occur?

6. h-astral configurations (n_3)

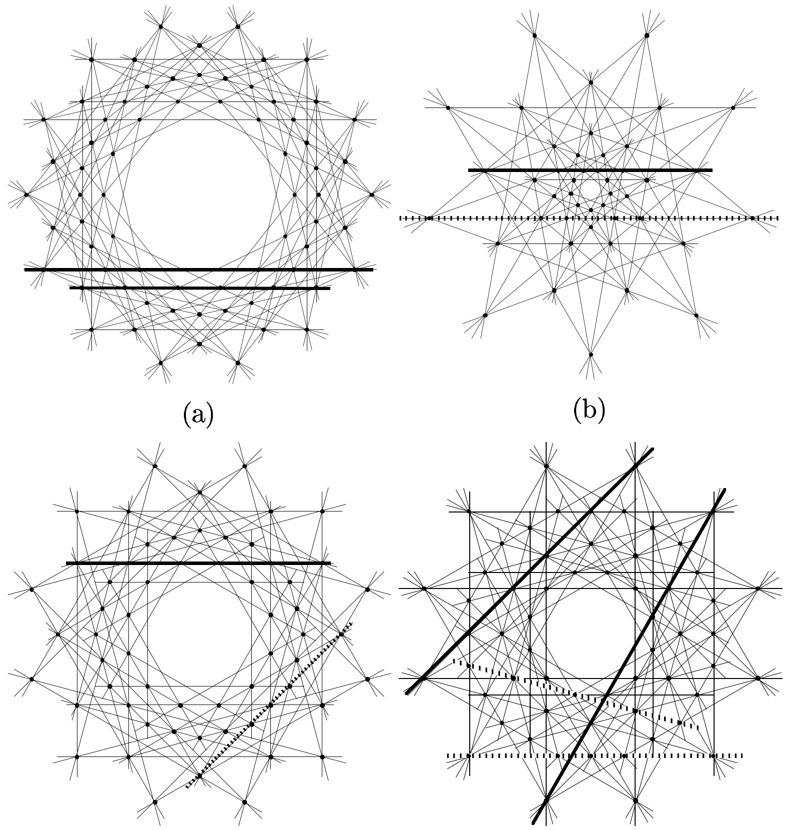
The study of these configurations is much less advanced, and promises to be more challenging than the investigation of the (n_4) configurations. There are two sources of the variety possible for h-astral (n_3) configurations. On the one hand, in many cases there is at least one parameter that can assume a continuum of different real values. On the other hand, if $h \geq 3$, a line of the configuration can contain points from either two or three different orbits. Even more than in the case of (n_4) configurations, the case $h = 2$ of astral configurations is radically different from h-astral with $h \geq 3$.

As mentioned above, the h-astral configurations (n_3) come in three varieties:

- *projectively h-astral*, that is configurations h-astral in the *extended Euclidean* (that is, *projective*) plane E^{2+}, but not in the Euclidean plane E^2 itself.
- *h-cyclic* (chiral), that is, configurations in E^2 with a cyclic symmetry group.
- *h-dihedral*, that is configurations E^2 with a dihedral symmetry group.

6.1. Examples of projectively astral configurations are shown in Figures 25 and 26. The first configuration in Figure 25 is a realization of the Pappus configuration; see Coxeter [**C5**]. A 3-astral realization of the Desargues configuration (10_3) in E^{2+} is given by Coxeter [**C4**, Figure 6]. It is clear that similar examples could be found for $h \geq 4$. At least for small h, the complete characterization of projectively astral configurations may be relatively simple but seems not to have been worked out.

6.2. *h-cyclic* configurations (n_3) are much more interesting. A few examples of cyclic configurations are shown in Figures 27 and 28. The first published graphical representations of h-cyclic configurations seem to be those of Visconti



(a) (b) (c) (d)

FIGURE 22. (a) A representation of the 5-astral configuration with symbol 14#(3,1,4,3,1,3,2,4,3,2); it has two orbits of lines incident with six points each. One line of each such orbit is emphasized. One orbit arises from the trivial configuration 14#(3,1,4,3,1,4). The other orbit arises from the sporadic configuration 14#(3,2,3,1,4,5). The points in the two outermost orbits are incident with six lines each. (b) A representation of the 5-astral configuration 9#(3,1,4,3,1,3,2,4,3,2). The heavily drawn line represents the orbit of lines through six points that arises from the trivial 3-cyclic configuration 9#(3,1,4,3,1,4). The dashed line arises from the 3-astral configuration with symbol 9#(3,1,3,4,3,2). However, this symbol does not arise from the original by the process in condition $(****)$. This seems to indicate that condition $(****)$ needs to be strengthened. (c) the representation of the 5-astral configuration 12#(3,1,4,3,1,3,2,4,3,2) has one orbit of lines incident with six points each, and one orbit of points incident with six lines. It also has one orbit of lines incident with five points, one of which is incident with five lines. (d) A representation of the 5-astral configuration 12#(4,3,1,4,3,4,3,1,4,3). It has two orbits of lines incident with six points each, and two orbits of lines incident with five points each; analogously for points.

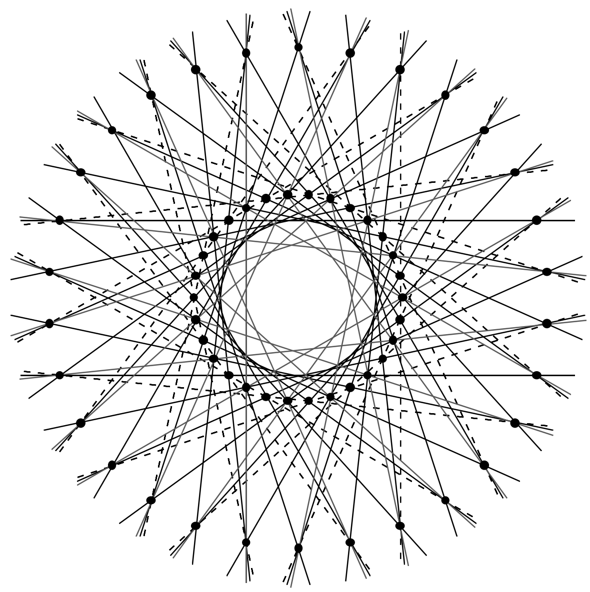


FIGURE 23. A 3-astral configuration (60_4).

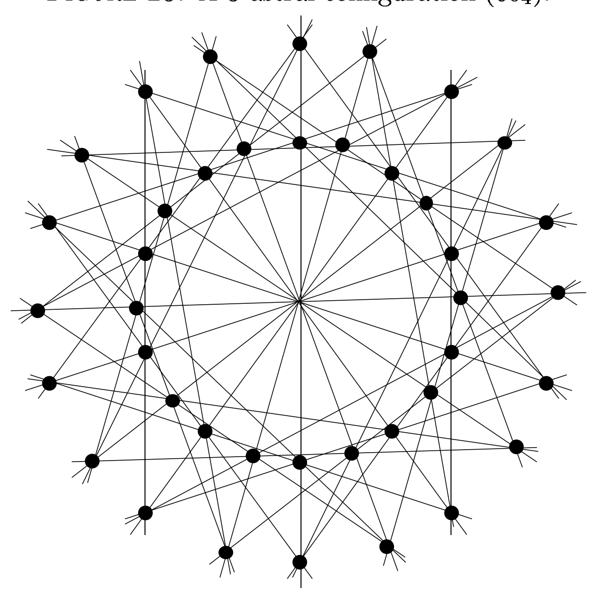


FIGURE 24. A 3-dihedral configuration (40_4), in which the points form two orbits and the lines form three orbits. It is constructed by starting with the vertices of a regular decagon, taking diagonals of span 3 and diameters. This is a configuration of type [3,4]. Taking a concentric copy, suitably rotated, one can add additional lines which yield a configuration (40_4). Here the angle of rotation is $\arccos((19\sqrt{5}-1)/44) = 19.464602895^0$.

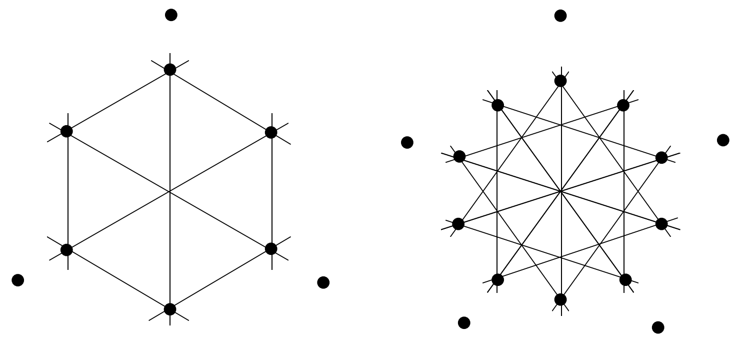


FIGURE 25. Examples of (n_3) configurations that are astral in the extended Euclidean plane E^{2+} but not contained in the Euclidean plane E^2 itself. The unattached dots indicate points at infinity.

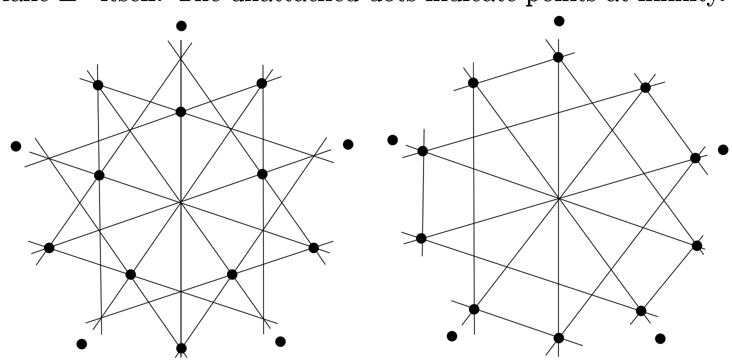


FIGURE 26. Two configuration (n_3) that are 3-astral in the projective plane E^{2+} but not in the Euclidean plane E^2. Note that the first one has only two orbits of lines, each containing points of all three orbits. The other has only two orbits of points, but three orbits of lines.

[**V1**]; a 4-cyclic (20_3) reproduced as Figure 9, and a 3-cyclic (30_3). It is ironic that Schönflies [**S2**] shows drawings of configurations (12_3) and (9_3) which could have been presented as 3-cyclic — but were drawn with non-regular polygons. This is the same Schönflies who a few years later determined the 230 discrete symmetry groups of the Euclidean 3-space! Zacharias [**Z1**] shows several examples of what are 2- and 3-cyclic configurations (n_3); he comments on their star-shaped appearance, and mentions that other such configurations may be formed — but does not discuss the symmetries as such, nor any general methods of construction.

It is obvious that each line of a 2-cyclic configuration must contain two points from the same orbit, and one from another orbit. The notation used in these examples will be explained later; it is a special case of the notation

$$m\#(b_1, b_2, \ldots, b_h; b_0; \lambda_1, \lambda_2, \ldots, \lambda_{h-2})$$

— or in a shorter symbol $m\#(b_1, b_2, \ldots, b_h; b_0)$ — for h-cyclic configurations of this kind. Here $n = hm$ and we have $h - 2$ real parameters λ_j besides $h + 1$ discrete ones b_j. Together these parameters lead to a quadratic equation for an additional

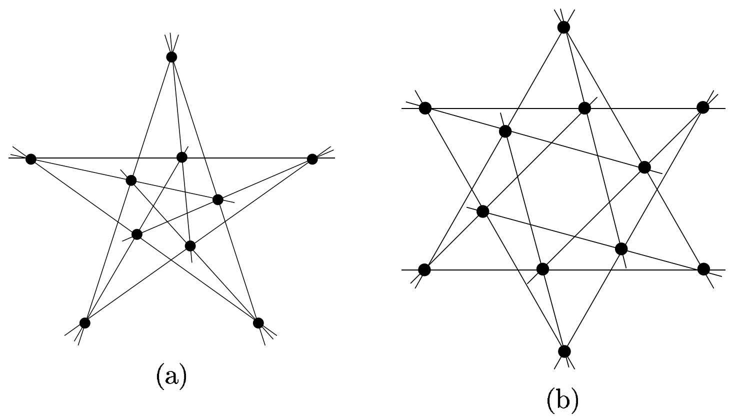


FIGURE 27. The two smallest 2-cyclic configurations (n_3): (a) (10_3) with symbol 5#(2,2;1) and (b) (12_3) with symbol 6#(2,2;1). The symbols are explained later in the text.

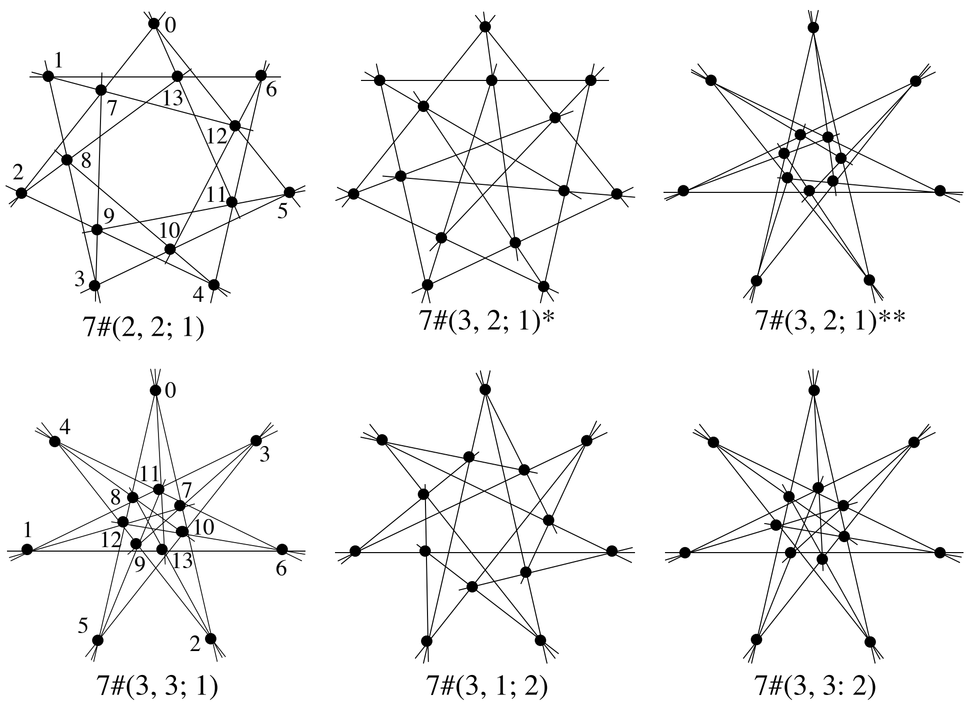


FIGURE 28. The six geometrically distinct 2-cyclic configurations (14_3), and their symbols. The four symbols without asterisks correspond to selfpolar configurations. The other two are polars of each other; since they have the same parameters, they are isomorphic as well. The two configurations in the left column are isomorphic; the labels indicate an isomorphism. Thus there are four non-isomorphic 2-cyclic configurations (14_3).

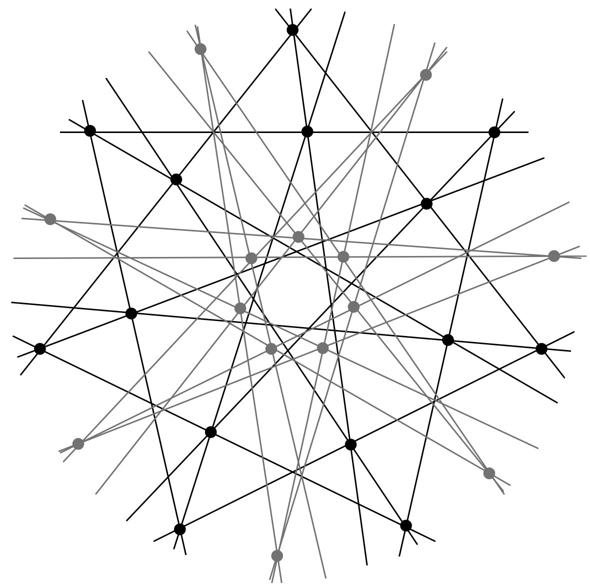


FIGURE 29. A pair of 2-astral configurations (14_3) polar to each other. Black: $7\#(3,2;1)^* = 7\#(3,2;1)$; gray: $7\#(3,2;1)^{**} = 7\#(2,3;4)$.

parameter. This equation can have 2, 1 or 0 real solutions — in the last case there are no corresponding configurations.

In particular, 2-cyclic configurations have no free parameters λ_j. The quadratic equation and the remaining real parameter are determined by the integer parameters $m\#(b_1, b_2; b_0)$. In case there are two distinct values for the real parameter, if necessary we append to the symbol $m\#(b_1, b_2; b_0)$ one or two asterisks * or **, to indicate whether the larger or the smaller value is used.

As pointed out in [**B8**], the dual of a configuration $m\#(b_1, b_2, \dots, b_h; b_0)$ is the configuration $m\#(b_h, b_{h-1}, \dots, b_1; b_1 + b_2 + \cdots + b_h - b_0)$. This is illustrated by the example in Figure 29. It should be noted that for 2-cyclic configurations, in the case the determining equation has two distinct real roots, the configurations $m\#(b, c; d)^*$ and $m\#(b, c; d)^{**}$ are duals of each other. In fact, as illustrated in Figure 29, they are related by a polarity.

Extensive experimental evidence shows that symbols $m\#(b, c; d)$ of representable 2-cyclic configurations must satisfy:

$$
\begin{aligned}
&0 < c \le b < m/2,\\
&c - b + 1 < 2d \le b + c,\\
&0 \ne d \ne c,\\
&2\cos(b\pi/m)\cos(c\pi/m) \le 1 + \cos((b + c - 2d)\pi/m).
\end{aligned}
$$

CONJECTURE 6.1. *The above conditions are necessary and sufficient for the existence of a 2-cyclic configuration* $m\#(b, c; d)$.

The 2-cyclic *selfpolar* configurations(n_3) are quite interesting; they come in three varieties. The first kind corresponds to symbols of the type $m\#(b, b; d)$, with

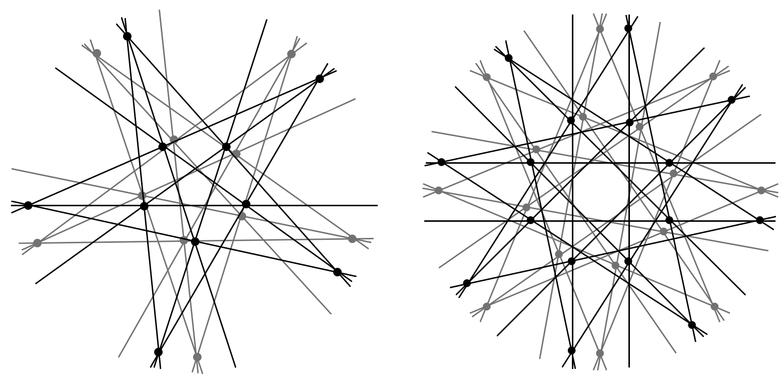


FIGURE 30. Negatively selfpolar configurations 5#(2,2;1) and 8#(3,3;2).

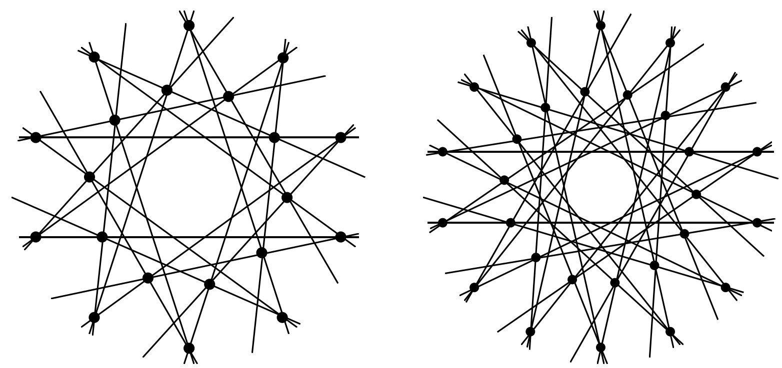


FIGURE 31. Positively selfpolar configurations 10#(4,2;3) and 13#(6,4;5), with b and c even.

$1 \le d < b$; they are *negatively selfpolar.* By this is meant that the polar of the configuration is a mirror image of the configuration itself. This is illustrated by the examples in Figure 30. The only other selfpolar configurations are the ones with symbol $m\#(b, c; d)$, where $b \ne c$ and $d = (b + c)/2$. They are *positively selfpolar*, that is, the polar is congruent to the original without reflection. But depending on the parity of b (and c), the polar either *coincides* with the original, or *differs* from it by a non-trivial rotation. These two possibilities are illustrated in Figures 31 and 32.

CONJECTURE 6.2. *There are no selfpolar 2-cyclic configurations* (n_3) *besides those described above.*

We turn now to explain the notation for h-cyclic configurations in which each line is incident with points of two orbits only; the remaining case will be described below. Our explanation is illustrated in Figure 33, using a 3-cyclic configuration (27_3) as an example.

As mentioned before, the symbol for an h-cyclic configuration (n_3), where $n = hm$, is of the form $m\#(b_1, b_2, \ldots, b_h; b_0; \lambda_1, \lambda_2, \ldots, \lambda_{h-2})$; the parameters are again determined by a *characteristic path.* The entries $b_1, \ldots, b_h$ are the spans of

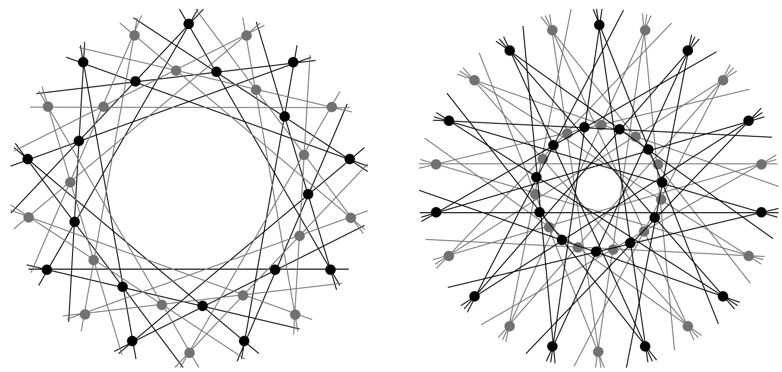
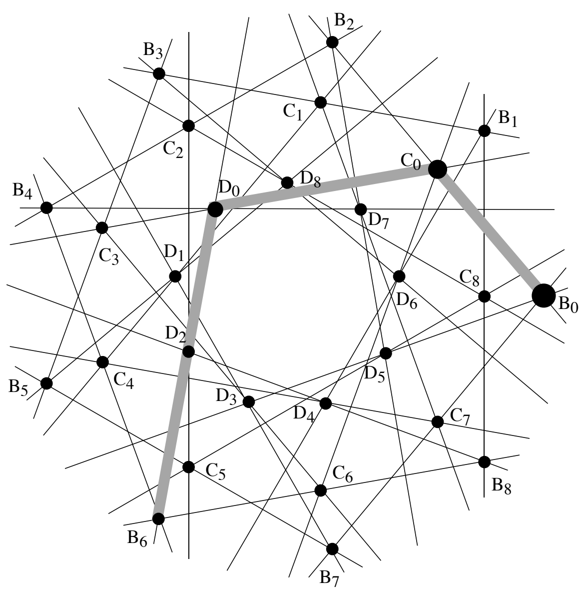


FIGURE 32. Positively selfpolar configurations 9#(3,1;2) and 11#(5,1;3), with b and c odd.



9#(2,3,2;6;0.5)

FIGURE 33. The characteristic path $B_0C_0D_0B_6$ is represented in the symbol by the first four parts; the first three integers are the spans of the respective diagonals, the fourth indicates the point of the first orbit at which the path ends. The last parameter indicates the ratio in which the segment B_0B_1 is divided by the point C_0. This parameter can be freely chosen (with some limits); the analogous parameters for the other two segments are then determined.

the diagonals in the different regular m-gons that are determined by the path; the diagonals are all oriented in the same way — clockwise or counterclockwise, and the real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{h-2}$ denote the ratios in which each diagonal determined by a segment of the path is divided by the endpoint of the segment. The path returns to the starting polygon, but not necessarily to the starting point of the path. The parameter b_0 indicates the vertex of the first polygon at which the characteristic path ends. These data lead to a quadratic equation for the ratio λ_{h-1} on the next-to-last segment; the ratio applicable to the last segment is then completely determined. Thus there are either two, or one, or no real geometric configurations corresponding to a given symbol. There are also possibilities of unintended incidences similar to the ones we encountered earlier, hence we are in general talking about *representations* of the symbols, rather than *realizations*. In case the parameters $\lambda_1, \lambda_2, \ldots, \lambda_{h-2}$ in a symbol $m\#(b_1, b_2, \ldots, b_h; b_0; \lambda_1, \lambda_2, \ldots, \lambda_{h-2})$ are not relevant or not known, we abbreviate the symbol to $m\#(b_1, b_2, \ldots, b_h; b_0)$.

The example in Figure 33 presents a 3-cyclic configuration with symbol $9\#(2, 3, 2; 6; .5)$. The points of the three orbits are denoted by B_j, C_j, D_j. The determination of the symbol is highlighted by the three-step characteristic path. Note that the ratio λ_1 can be chosen freely, and in the illustration it was taken as $\lambda_1 = 0.5$. Once the first $h - 2 = 1$ ratios λ_j are chosen, the last ratio λ_{h-1} (determining the position of the point of last orbit on the penultimate diagonal) is determined by a quadratic equation. (For details see [**B8**].) In the illustration $\lambda_{h-1} = \lambda_2$ is about 2/3. Naturally, the symbol is not unique since it depends, besides the λ_j's for $h \geq 3$, on the orbit of the starting point, and on the orientation chosen. The influence of the parameter λ_{h-2} is illustrated in Figure 34.

Using symbols like $u, v, w, \ldots$ for elements of the different orbits of points, we can say that the h-cyclic configurations considered so far have lines of type $\{u, u, v\}, \{v, v, w\}, \ldots$. But other possibilities exist in which the incidences of lines with orbits of the points are different. For example, in case $h = 3$, it is possible to have three orbits of lines, all three of the type $\{u, v, w\}$, or else, one of the type $\{u, v, w\}$ and the other two of types $\{u, v, v\}$ and $\{u, w, w\}$. Three examples of the former variety are shown in Figure 35, while examples of the second kind are illustrated in Figure 36; the rightmost diagram in Figure 3 shows the $(9_3)_3$ configuration, which is of this kind. A notation for the configurations in Figure 35 is explained in the caption; no notation has been proposed for the kind of configurations shown in Figure 36. No additional details about either of the kinds are available as of this writing.

h-cyclic configurations in which lines of one or more orbits are incident with points of three orbits have not been studied at all.

6.3. *h-dihedral configurations* are also largely uninvestigated. One difference in comparison with h-cyclic configurations is that even for $h = 2$, isomorphic 2-hedral configurations admit a real-valued parameter. This is illustrated in Figure 37 by a family of 2-hedral configurations (12_3); the configurations in each row are polars of each other. Examples of other kinds of 3-dihedral configurations are shown in Figures 38 and 39.

There are many open problems related to h-astral configurations. In most cases the information available at this time does not lead to any specific conjectures. Here are a few examples.

- Are non-Hamiltonian h-astral configurations (n_3)?

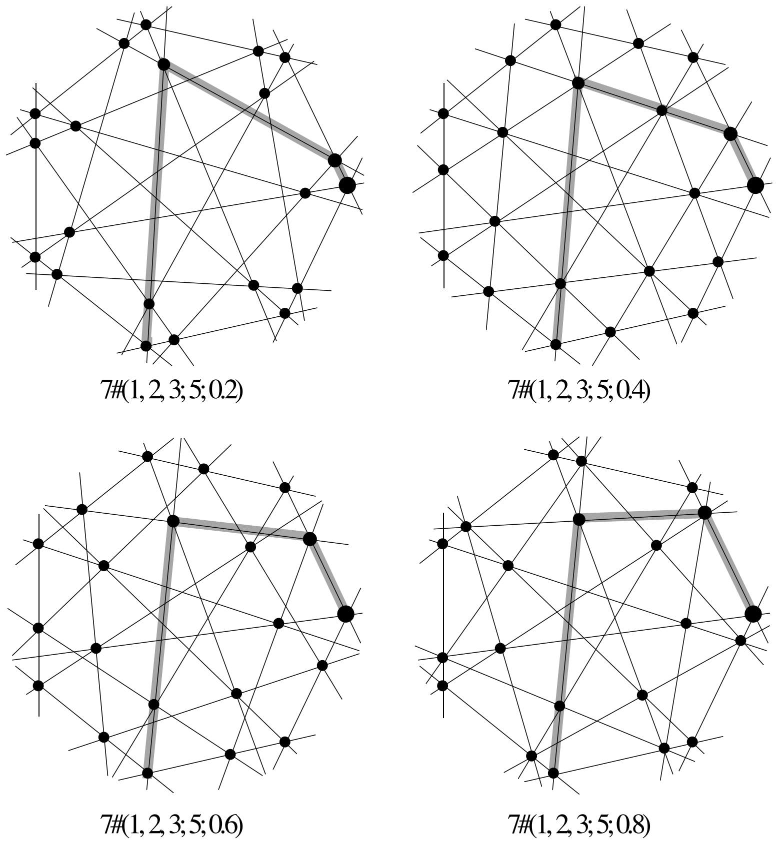
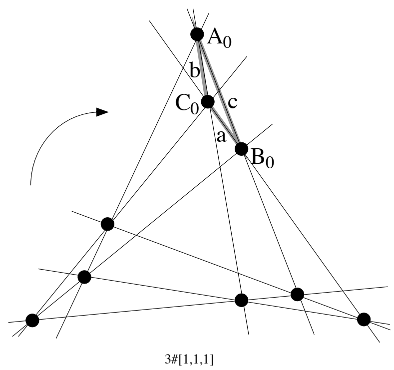
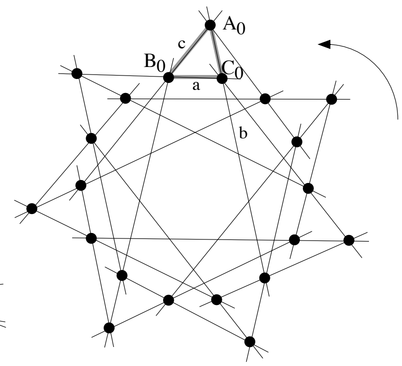


FIGURE 34. The dependence of the configuration 7#(1,2,3;5;λ_1) on the parameter λ_1. For a value of λ_1 somewhat smaller than 0.8 the representation would not be a realization. The last diagram can be interpreted as nearly illustrating this case.

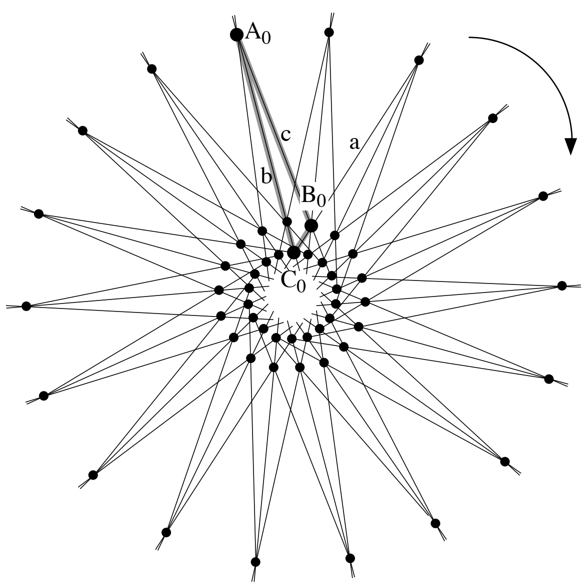
- Determine the number of different isomorphism classes of h-cyclic configurations (n_3), at least for $h = 2$. What relations exist among the parameters of isomorphic h-cyclic configurations?
- Are there combinatorial configurations (n_3) that can be represented by h-cyclic configurations for different values of h?
- Characterize self-dual polydihedral configurations.
- Does every astral configuration (n_4) contain an astral subconfiguration (n_3) ?
- Which 2-cyclic (or 2-dihedral) configurations (n_3) are only representations (not realizations) of the underlying combinatorial configurations?



(a) 3#[1,1,1]



(b) 7#[1,4,2]



(c) 18#[2,-1,3]

FIGURE 35. Three examples of 3-cyclic configurations, in which each line is incident with one point in each of the three point orbits. The symbol near each can be used as a description. After choosing a triangle (gray lines) with vertices in the three orbits, the numbers in brackets indicate which of the points A_i, B_j, C_k is one of the lines a, b, c, respectively. (a) is a realization of the Pappus configuration $(9_3)_1$. The configuration in (b) has as its Levi graph the only cyclic 3-valent graph with 42 vertices; it appears in the Foster census **[F1]**.

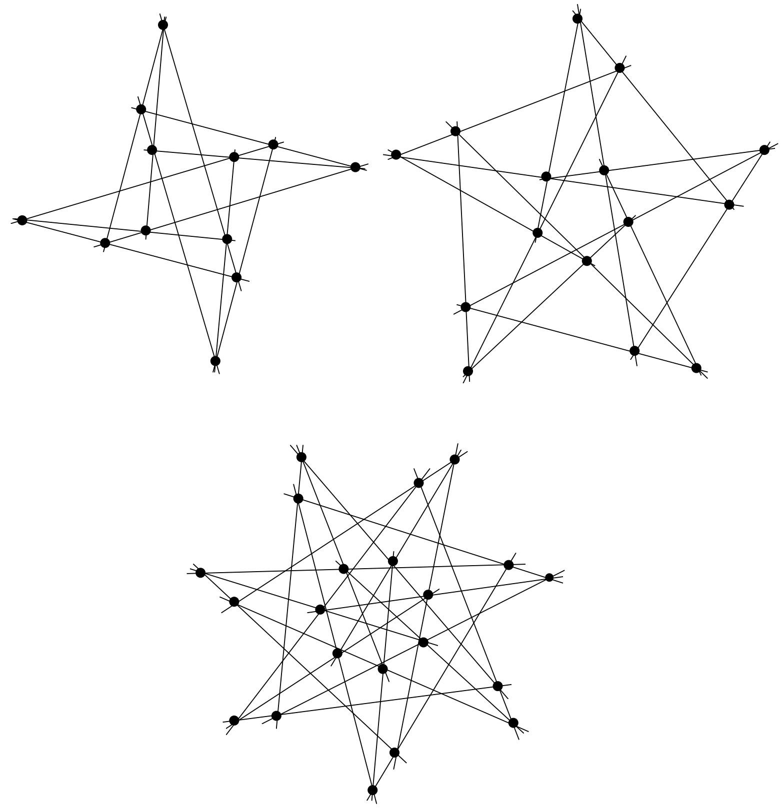


FIGURE 36. In these 3-cyclic configurations there is one orbit of lines each of which is incident with one point of each of the three point orbits. There are two orbits of lines each of which is incident with points from only two point orbits each.

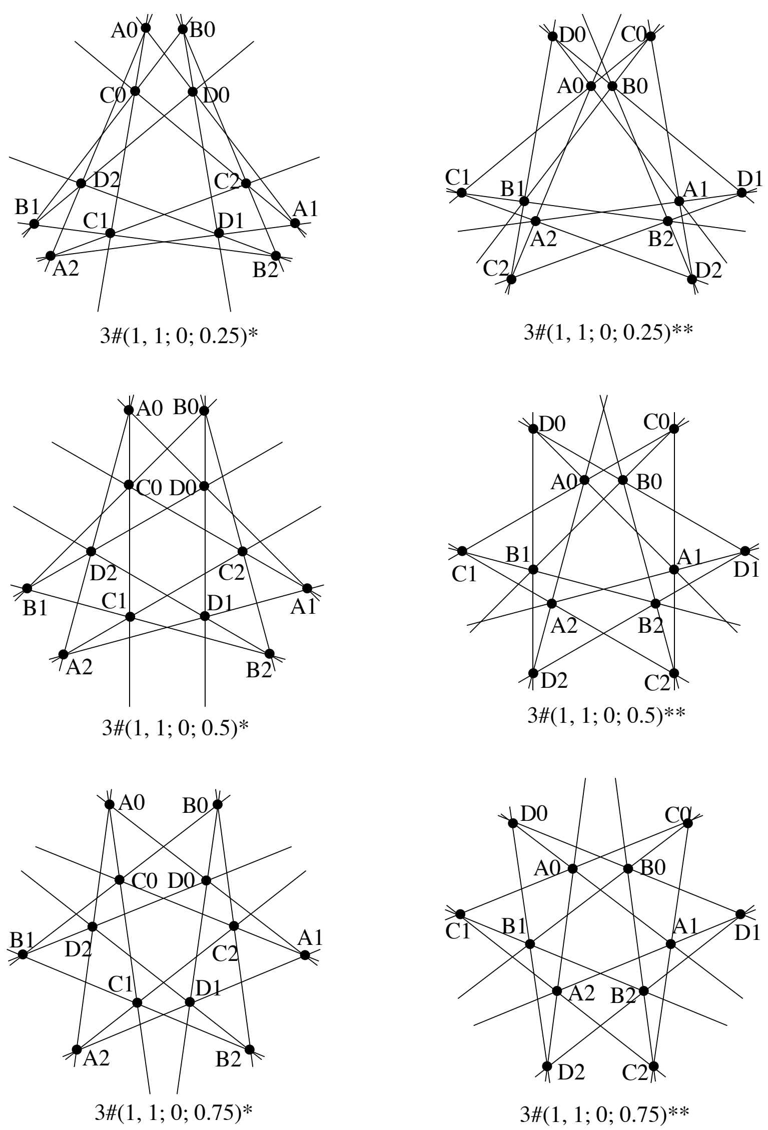
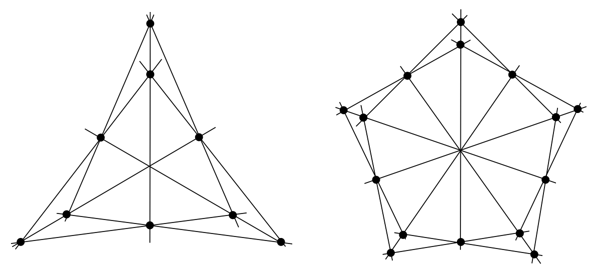


FIGURE 37. Pairs of mutually polar 2-dihedral configurations (12_3), all isomorphic, illustrating the dependence of the appearance of the realization on the value of a convenient parameter.



(a) (b)

FIGURE 38. Two 3-dihedral configurations, (a) is another realization of the Pappus configuration (9_3); this is the smallest 3-dihedral configuration. (b) A (15_3) configuration.

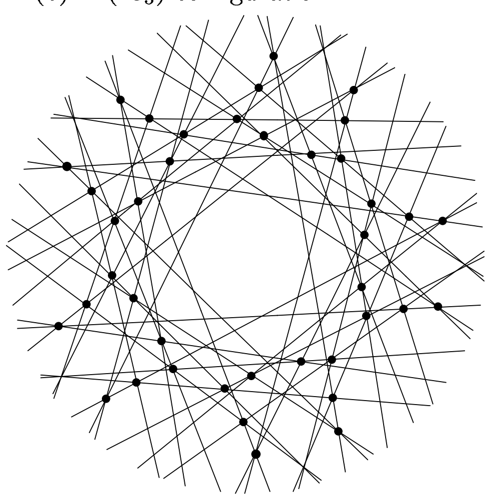


FIGURE 39. A 5-dihedral configuration (40_3). (Courtesy of L. Berman)

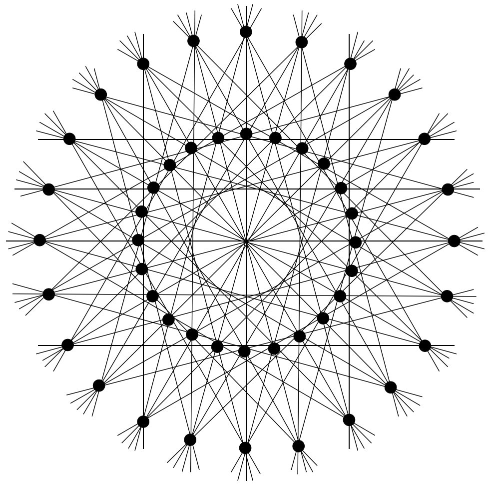


FIGURE 40. A 3-astral configuration $(48_5, 60_4)$. Adding the 12 points of infinity in the direction of the lines of that configuration yields a (60_5) configuration 3-astral in the extended Euclidean plane E^2.

7. Other astral configurations

It is easy to show that the only configurations (n_4) that are astral in the projective (extended Euclidean) plane E^{2+} are also astral in the Euclidean plane. The same goes for astral configurations (n_6) — but vacuously: there exist no astral configurations (n_6) (see Berman [**B2**]). However, the situation is different for astral configurations (n_5). Starting with two concentric copies of the astral configuration (24_4) shown in Figure 12, rotated by $\pi/24$ with respect to each other, and adding 12 diametral lines, an astral configuration $(48_5, 60_4)$ is obtained. By adding 12 points at infinity, a (60_5) configuration astral in the extended Euclidean plane E^{2+} results, see Figure 40. A similar construction works with all astral configurations (n_4). The astral configuration (50_5) indicated in Figure 41 is obtained by a variant of that construction, possible in this case. These diagrams are the first published depictions of any (n_5) configuration, as well as of any configuration of type $[5, 4]$. It may be conjectured that the (50_5) configuration in Figure 41 is the smallest (n_5) configuration astral in the extended Euclidean plane E^{2+}. However, we have:

CONJECTURE 7.1. *There exist no (n_5) configurations astral in the Euclidean plane E^2.*

Nothing seems to be known about the existence or nonexistence of configurations (n_5) or (n_6) that are h-astral in the Euclidean plane E^2 for $h \geq 4$.

Concerning geometric configurations of type $[q, k]$ that are not equinumerous (that is, for which $q \neq k$) very little information is available. Data on combinatorial configurations of this kind can be found in [**G4**] and [**G5**].

It is well known that for each integer $r \geq 3$ there exists a combinatorial configuration $((4r)_3, (3r)_4)$, and these are the only possible ones of type $[3, 4]$. There is no geometric configuration $(12_3, 9_4)$, but for every $r \geq 4$ there exist geometric configurations $((4r)_3, (3r)_4)$ astral in E^2. Typical examples are shown in Figure 42.

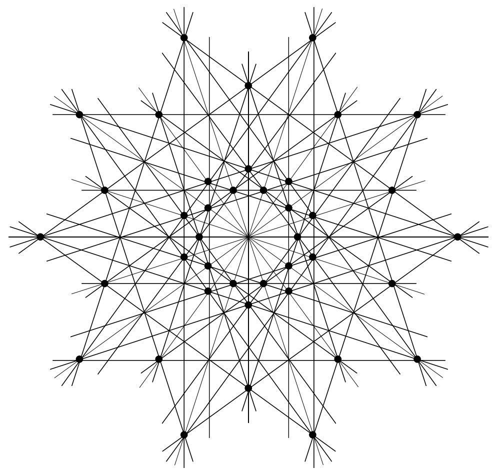


FIGURE 41. Without the lines through the center, this is the (40_4) configuration with symbol 10#(4,3,1,2,3,4,2,1). Including these lines gives a 3-astral configuration $(40_5, 50_4)$. With the addition of ten ideal points we obtain a configuration (50_5) that is 3-astral in the extended Euclidean plane E^{2+}.

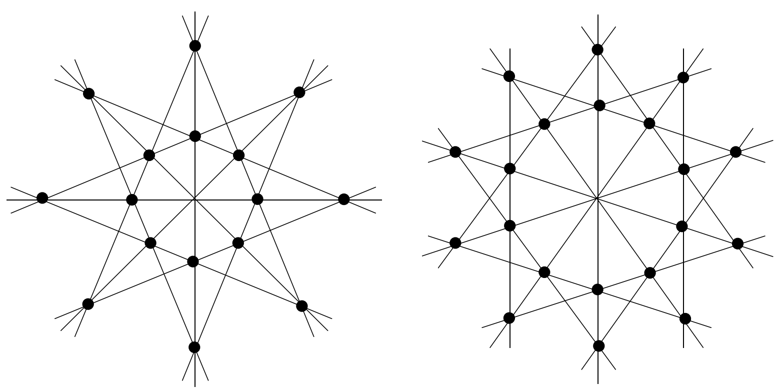


FIGURE 42. The smallest geometric configuration of type [3,4], astral in E^2.

Configurations $(12_4, 16_3)$, have been investigated (at least in special cases) for close to two centuries. Details and references are available in [**G3**], where it is stated that there are 574 combinatorial configurations $(12_4, 16_3)$. It is not known how many of them are geometric. Polars of the configurations in Figure 42 are configurations of type $[4, 3]$ astral in the extended Euclidean plane E^{2+}. It is not known whether any are astral in E^2.

For each integer $r \geq 4$ there exists a combinatorial configuration $((5r)_3, (3r)_5)$; no other configurations of type $[3, 5]$ are possible. There is no geometric configuration $(20_3, 12_5)$, but for every $r \geq 4$ there exist geometric configurations $((5r)_3, (3r)_5)$

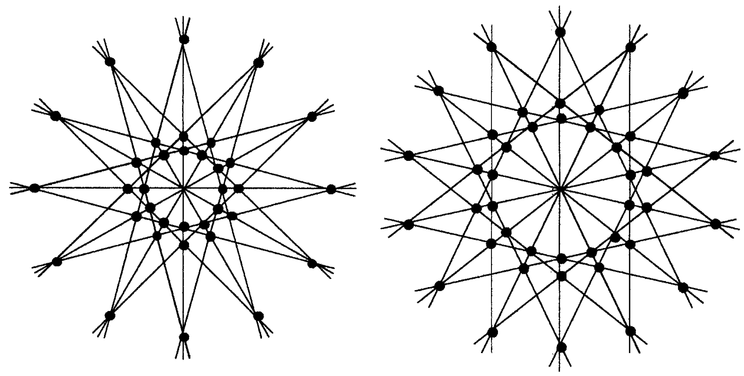


FIGURE 43. Configurations $(36_3, 18_6)$ and $(42_3, 21_6)$ astral in the Euclidean plane.

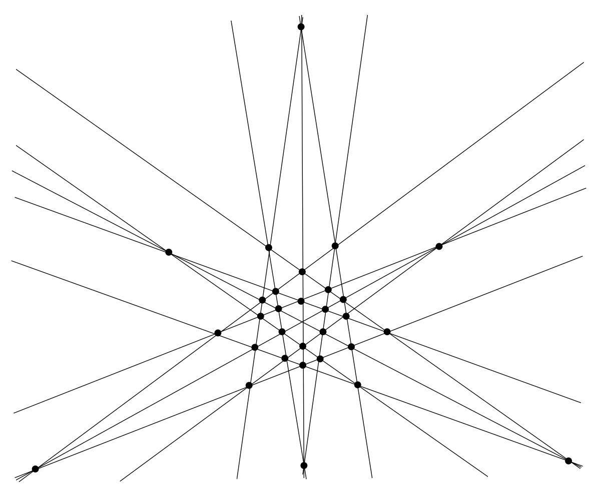


FIGURE 44. A configuration $(30_3, 15_6)$. Deleting the six outermost points and adding the point at the center yields a configuration $(25_3, 15_5)$.

astral in E^2; their construction is analogous to the one of configurations of type [3, 4] in Figure 42.

It can be shown that combinatorial configurations of type [3, 6] exist if and only if the parameters are $((2r)_3, r_6)$ with $r \geq 13$. For all $j \geq 6$ there exist geometric configurations $((6j)_3, (3j)_6)$ astral in the Euclidean plane; typical examples are shown in Figure 43. It can be shown that astral configurations $((2r)_3, r_6)$ do not exist if r is not a multiple of 2 or a multiple of 3; however, no example is known in which r is not a multiple of 3. It is also not known whether there exist any geometric configurations $(28_3, 14_6)$, but a non-astral configuration $(30_3, 15_6)$ exists and is indicated in Figure 44.

This essentially exhausts the current information available about astral and other geometric configurations. It clearly leaves many kinds of open problems, ranging from very elementary to highly technical. It is my hope that the present exposition will contribute to a renewed interest in geometric configurations, and in their generalizations (such as configurations of pseudolines).

References

[B1] L. W. Berman, *A characterization of astral (n_4) configurations.* Discrete Comput. Geom. 26(2001), no. 4, 603–612.

[B2] L. W. Berman, *Even astral configurations.* Electron. J. Combin. 11(2004), Research Paper 37, 23 pp. (electronic).

[B3] A. Betten and D. Betten, *Tactical decompositions and some configurations v_4.* J.Geom. 66(1999), 27–41.

[B4] A. Betten, G. Brinkmann and T. Pisanski, *Counting symmetric configurations v_3.* Discrete Appl. Math. 99(2000), 331–338.

[B5] M. Boben, *Uporaba teorije grafov pri kombinatori/nih in geometri/nih konfiguracijah.* (In Slovenian) [The use of graph theory in the study of combinatorial and geometric configurations]. Ph.D. thesis, University of Ljubljana, November 2003.

[B6] M. Boben, *Irreducible (v_3) configurations and graphs.* (To appear)

[B7] M. Boben, B. Grünbaum, T. Pisanski and A. Žitnik, *Small triangle-free configurations of points and lines.* Dept. of Math, Univ. of Ljubljana, Preprint series 42(2004), #938.

[B8] M. Boben and T. Pisanski, *Polycyclic configurations.* Europ. J. Combin. 24(2003), 431–457.

[B9] J. Bokowski and B. Sturmfels, *Computational Synthetic Geometry.* Lecture notes in Mathematics #1355, Springer, New York 1989.

[B10] G. Bol, *Beantwoording van prijsvraag no. 17.* Nieuw Archief voor Wiskunde 18(1931), 14–66 (1933).

[B11] G. Brunel, *Polygones á autoinscription multiple.* Proc. Verb. Séances Soc. Sci. phys. nat. Bordeau, 1897/98, pp. 43–46.

[B12] W. Burnside, *On the Hessian configuration and its connection with the group of 360 plane collineations.* Proc. London Math. Soc. (2) 4(1907), 54–71.

[C1] H. G. Carstens, T. Dinski and E. Steffen, *Reduction of symmetric configurations n_3.* Discrete Appl. Math. 99(2000), 401–411.

[C2] H. S. M. Coxeter, *Configurations and maps.* Reports of a Math. Colloq. (2) 8(1949), 18–38.

[C3] H. S. M. Coxeter, *Self-dual configurations and regular graphs.* Bull. Amer. Math. Soc. 56(1950), 413–455. (= Twelve Geometric Essays, Southern Illinois Univ. Press, Carbondale, Il 1968 = The Beauty of Geometry, Dover, Mineola, NY 1999, pp. 106–149.

[C4] H. S. M. Coxeter, *Desargues configurations and their collineation groups.* Math. Proc. Cambridge Philos. Soc., 78(1975), 227–246.

[C5] H. S. M. Coxeter, *The Pappus configuration and the self-inscribed octagon. I, II, III.* Nederl. Akad. Wetensch. Proc. Ser. A 80 = Indag. Math. 39(1977), pp. 256–269, 270–284, 285–300.

[C6] H. S. M. Coxeter, *My graph.* Proc. London Math. Soc. (3) 46(1983), 117–136.

[D1] R. Daublebsky von Sterneck, *Die Configurationen* 11_3. Monatshefte Math. Phys. 5(1894), 325–330 + 1 plate.

[D2] R. Daublebsky von Sterneck, *Die Configurationen* 12_3. Monatshefte Math. Phys. 6(1895), 223–255 + 2 plates.

[D3] R. Daublebsky von Sterneck, *Über die zu den Configurationen* 12_3 *zugehörigen Gruppen von Substitutionen.* Monatshefte Math. Phys. 14(1903), 254–260.

[D4] H. L. Dorwart and B. Grünbaum, *Are these figures oxymora?* Mathematics Magazine 65(1992), 158–169.

[F1] R. M. Foster, *The Foster Census.* Charles Babbage Research Centre, Winnipeg, MB, 1988, viii+240 pp.

[G1] H. Gropp, *Il methodo di Martinetti (1887) or Configurations and Steiner systems S(2, 4, 25).* Ars Combinatoria 24B(1987), 179–188.

[G2] H. Gropp, *On the existence and nonexistence of configurations n_k.* J. Combinatorics, Information and System Science 15(1990), 34–48.

[G3] H. Gropp, *The construction of all configurations* $(12_4, 16_3)$. Fourth Czechoslov. Symp. on Combinatorics, Graphs and Complexity, J. Nešetřil and M. Fiedler, eds. Elsevier 1992.

[G4] H. Gropp, *Non-symmetric configurations with deficiencies 1 and 2. In "Combinatorics '90"*, A. Barlotti et al., eds. Elsevier 1992, pp. 227–239.

[G5] H. Gropp, *Nonsymmetric configurations with natural index.* Discrete Math. 124(1994), 87–98.

[G6] H. Gropp, *Configurations and their realizations.* Discrete Math. 174(1997), 137–151.

[G7] H. Gropp, *On combinatorial papers of König and Steinitz.* Acta Applicandae Math. 52(1998), 271–276.

[G8] H. Gropp, *Configurations between geometry and combinatorics.* Discrete Appl. Math. 138(2004), 79–88.

[G9] B. Grünbaum, *Convex Polytopes.* Wiley, New York 1967. 2nd ed. Springer, New York 2003.

[G10] B. Grünbaum, *Astral* (n_k) *configurations.* Geombinatorics 3(1993), 32–37.

[G11] B. Grünbaum, *Astral* (n_4) *configurations.* Geombinatorics 9(2000), 127–134.

[G12] B. Grünbaum, *Connected* (n_4) *configurations exist for almost all* n *– an update.* Geombinatorics 12(2002), 15–23.

[G13] B. Grünbaum, *A 3-connected non-Hamiltonian configuration* (50_3). (In preparation.)

[G14] B. Grünbaum and J. F. Rigby, *The real configuration* (21_4). J. London Math. Soc. (2) 41(1990), 336–346.

[H1] D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie.* Springer, Berlin 1932. English translation: *Geometry and the Imagination*, Chelsea, New York 1952.

[K1] F. Klein, *Über die Transformationen siebenter Ordnung der elliptischen Funktionen.* Math. Ann. 14(1879), 428–471.

[K2] S. Kantor, *Über eine Gattung von Configurationen in der Ebene und im Raume.* Wien. Ber. LXXX(1879), 227.

[K3] S. Kantor, *Über die Configurationen (3,3) mit den Indices 8, 9 und ihren Zusammenhang mit den Curven dritter Ordnung.* Wien. Ber. LXXXIV(1881), 915–932.

[K4] S. Kantor, *Die Configurationen* $(3,3)_{10}$. Wien. Ber. LXXXIV(1881), 1291–1314 + plate.

[K5] D. König, *Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre.* Math. Ann. 77(1916), 453–465.

[L1] R. Laufer, *Die nichkonstruierbare Konfiguration* (10_3). Math. Nachrichten 11(1954), 303–304.

[L2] F. Levi, *Geometrische Konfigurationen.* Hirzel, Leipzig 1929.

[L3] F. W. Levi, *Finite Geometrical Systems.* University of Calcutta, Calcutta, 1942.

[L4] G. Loria and E. K. Lampe, Review of [M1]. Jahrbuch Fortschr. Math. 19(1887), 587–589.

[M1] V. Martinetti, *Sulle configurazioni piane* μ_3. Annali di matematica pura ed applicata (2) 15(1887), 1–26.

[M2] E. Merlin, *Sur les configurations planes* n_4. Bull. Cl. Sci. Acad. Roy. Belg. 1913, 647–660.

[M3] A. F. Möbius, *Kann von zwei dreiseitigen Pyramiden eine jede in Bezug auf die andere um- und eingeschreiben zugleich heisen?* J. reine angew. Math. 3(1828), 273–278 = Gesammelte Werke 1(1885), 439–446.

[P1] W. Page and H. L. Dorwart, *Numerical patterns and geometrical configurations.* Math. Magazine 57(1984), 82–92.

[P2] T. Pisanski, Talk at the Ein Gev conference, 2000.

[P3] B. Poonen and M. Rubinstein, *The number of intersection points made by the diagonals of a regular polygon.* SIAM J. Discrete Math. 11(1998), 135–156.

[R1] T. Reye, *Geometrie der Lage. I.* 2nd ed. (1876).

[R2] T. Reye, *Das Problem der Configurationen.* Acta Math. 1(1882), 93–96.

[R3] J. F. Rigby, *Multiple intersections of diagonals of regular polygons, and related topics.* Geom. Dedicata 9(1980), 207–238.

[R4] C. Rodenberg, Review of [K4]. Jahrbuch Fortschr. Math. 13(1881), 460.

[S1] A. Schönflies, *Über einige ebene Configurationen und die zugehörigen Gruppen von Substitutionen.* Nachr. Ges. Wiss. Göttingen 1887, 410–417.

[S2] A. Schönflies, *Über die regelmässigen Configurationen* n_3. Math. Ann. 31(1888), 43–69.

[S3] H. Schroeter, *Über lineare Konstruktionen zur Herstellung der Konfigurationen* n_3. Nachr. Ges. Wiss. Göttingen 1888, 193–236.

[S4] H. Schroeter, *Über die Bildungsweise und geometrische Construction der Configurationen* 10_3. Nachr. Ges. Wiss. Göttingen 1889, 239–253.

[S5] H. Schubert, Review of [K3]. Jahrbuch Fortschr. Math. 13(1881), 460.

[S6] E. Steinitz, *Über die Construction der Configurationen* n_3. Ph.D. Thesis, Breslau 1894.

[S7] E. Steinitz, *Über die Unmöglichkeit, gewisse Configurationen n_3 in einem geschlossenen Zuge zu durchlaufen.* Monatshefte Math. Phys. 8(1897), 293–296.

[S8] E. Steinitz, *Konfigurationen der projektiven Geometrie.* Encyklopädie der math. Wissenschaften, Vol. 3 (Geometrie), Part IIIAB5a(1910), pp. 481–516 .

[S9] E. Steinitz, *Über Konfigurationen.* Archiv Math. Phys., 3rd Ser., 16(1910), 289–313.

[S10] E. Steinitz and E. Merlin, *Configurations.* French translation of [S8], incomplete. Encyclopédie des Sciences Mathématiques, edition française. Tome III, Vol. 2(1913), pp. 144–160.

[S11] R. Sternfeld, D. Koster, D. Kiel and R. Killgrove, *Self-dual confined configurations with ten points.* Ars Combinat. 67(2003), 37–63.

[S12] B. Sturmfels and N. White, *Rational realizations of* 11_3- *and* 12_3*-configurations.* In "*Symbolic Computations in Geometry*", by H. Crapo, T. F. Havel, B. Sturmfels, W. Whiteley and N. L. White, IMA Preprint Series #389, Univ. of Minnesota 1988, pp. 92–123.

[S13] B. Sturmfels and N. White, *All* 11_3- *and* 12_3*-configurations are rational.* Aequat. Math. 39(1990), 254–260.

[T1] E. Togliatti, Review of [Z2]. Zentralblatt Math. 43(1952), p. 358.

[V1] E. Visconti, *Sulle configurazioni piane atrigone.* Giornale di Matematiche di Battaglini 54(1916), 27–41.

[Z1] M. Zacharias, *Streifzüge im Reich der Konfigurationen: Eine Reyesche Konfiguration* (15_3)*, Stern- und Kettenkonfigurationen.* Math. Nachrichten 5(1951), 329–345.

[Z2] M. Zacharias, *Die ebenen Konfigurationen* (10_3)*.* Math. Nachrichten 6(1951), 129–144.

[Z3] M. Zacharias, *Bemerkung zu meiner Arbeit: Die ebenen Konfigurationen* (10_3)*.* Math. Nachrichten 12(1954), p. 256.

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Correction

The sentence in the middle of the first paragraph of subsection 3.2, and the corresponding diagram in Figure 7 are incorrect. They should be replaced by the following:

... It replaces two "parallel" lines (that is, lines no common point) such that the points A and D are not contained in any line of the configuration, by the three lines through pairs AD, BC and EF of these points and a new point O. ...

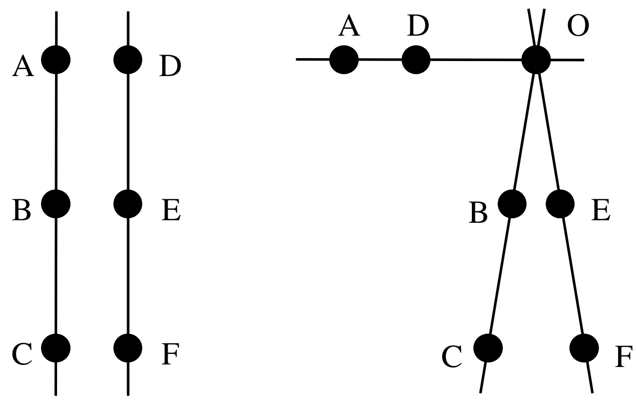

Figure 7. The Martinetti transformation.