ARE PRISMS AND ANTIPRISMS REALLY BORING ? (Part 3)

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Dedicated to V. G. Boltyanski, on his eightieth birthday

1. Introduction. In the second part of this paper [4] we considered the unjustifiably low reputation of antiprisms, and provided illustrations of the different approaches to these polyhedra. This part was devoted to the case of 3–dimensional polyhedra; here we shall deal with higher dimensions, mostly with 4-dimensional convex antiprisms. In order to make this part more selfcontained, here is the definition we have adopted:

For $d \geq 3$, a *convex d-dimensional antiprism* P with *bases* P_1 and P_2 is the convex hull of convex $(d-1)$-polytopes P_1 and P_2 provided:

(i) P_1 and P_2 are situated in distinct parallel hyperplanes, and are dual to each other under a mapping f;

(ii) the only other facets (that is, $(d-1)$-dimensional faces) of P are the convex hulls of faces F_1 and F_2 of P_1 and P_2, which correspond to each other under f.

For this definition see [3, p. 66] or Broadie [2]. If f is induced by polarity in a (d-1)-dimensional sphere we shall say that P is a *(p)-antiprism.*

In the next section we shall first formulate our principal result, that every polyhedron (that is, 3-polytope) is isomorphic (has the same combinatorial type) to one of the bases of a 4-dimensional (p)-antiprism. Sections 3 and 4 present the ingredients of the proof together with appropriate examples and some historical notes. Section 5 deals with various comments and open problems.

2. The main theorem. In the first part [4] of this paper we have seen that every polygon can be one of the bases of a 3-dimensional antiprism, but that one cannot insist that the other basis be a polar of the first one. However, it is not known whether every 3-dimensional convex polyhedron can serve as a basis of a 4-dimensional antiprism. On the other hand, a negative answer is clear if one insists on (p)-antiprisms. To see examples, one could modify the 2-dimensional ones from [4], but it is more useful to recall the following result of Broadie [2]:

Theorem 1. A convex (d-1)-polytope P and a polar P* of P can be the bases of a d-dimensional (p)-antiprism if and only P has the following property:

(*) There exists a point O such that for every face F (of *any* dimension) of P the orthogonal projection of O into the affine hull of F is in the relative interior of F itself.

As an illustration, consider the octahedron shown in Figure 1, obtained by cutting off two edges of a cube in a suitable way. On the other hand, the polyhedron in Figure 1 is a hexagonal prism; hence it is isomorphic to the regular hexagonal prism, which does satisfy Broadie's criterion. This leads to the question:

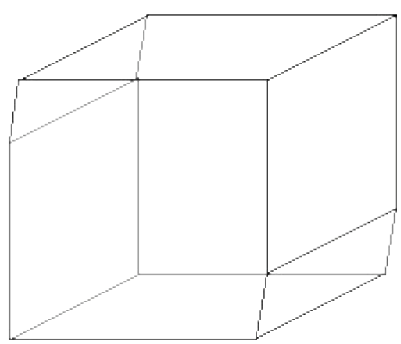

Figure 1. This polyhedron has no polar such that the two are the bases of a 4-dimensional (p)-antiprism. This follows from Broadie's criterion, since no point can serve as O: The slanted facet at left requires O to be near the top, while the other slanted facet requires it to be near the bottom.

Problem 1. Is every (d-1)-polytope isomorphic to a polytope P satisfying condition (*), such that P can be used with a suitable polar as bases of a d-dimensional (p)-antiprism?

As mentioned above, we know from [4] that the answer is affirmative for d = 3; the main result of this note is that Problem 1 has an affirmative solution for d = 4 as well.

Theorem 2. Every 3-polytope P is isomorphic to a polytope Q such that Q and a suitable polar Q^ of Q are bases of a 4–dimensional (p)-antiprism.

A proof of Theorem 2 will be sketched in the next section, after the introduction of some concepts needed to formulate the result on which the proof is based.

3. Midscribed polyhedra. A convex polyhedron P is said to be *midscribed* about a sphere S provided every edge of P is tangent to S. In such a situation we shall also say that S is the *midsphere* of P. The point at which an edge touches the midsphere is called the *tangency point* of that edge.

It is immediate that if a polyhedron P has a midsphere S, then each facet F (that is, 2-dimensional face) of P intersects S in a circle, which is the inscribed circle C of F. In other words, each facet F of P has an incircle C, and C has all edges of F as tangents at the tangency points of the edges. This implies that the incircles of adjacent facets meet are the tangency point of the edge common to these facets. Therefore the incircles of all facets of P form a *circle packing*. In general, a circle packing is a collection of circles (in our case, incircles of the facets of a polyhedron) that have pairwise one common point if and only if they correspond to adjacent facets. Note, however, that the facets of a polyhedron may have incircles that form a circle packing without the polyhedron having a midsphere. An example is shown in Figure 2.

A polyhedron is said to be of *midscribable type* if and only if it is isomorphic to a midscribable polyhedron. The question whether every polyhedron is of midscribable type seems to have been posed first as Problem 2 of [5]. For the proof of Theorem 2 we need the following surprising result: *Every convex polyhedron is of midscribable type*. More specifically, we have:

Theorem 3. For every polyhedron P there is a polyhedron Q isomorphic to P that is midscribed to a sphere S and is such that the centroid (center of gravity) of the tangency points coincides with the center of S. This polyhedron Q is determined uniquely in the sense that all polyhedra satisfying the same conditions are congruent to Q.

An illustration of Theorem 3 is shown in Figure 3. The polyhedron at left is isomorphic to the one at right, which is midscribed to a sphere; the tangency points are indicated by the solid dots.

We shall not give a proof of Theorem 3. Instead, in Section 4 details of its history will be presented, together with references to ways of establishing its validity.

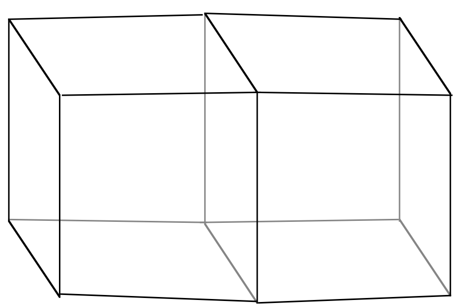

Figure 2. This polyhedron is formed as the union of two cube-like polyhedra, each of which has one facet smaller than the other, and the remaining four facets are trapezoids of appropriate shape, so that each facet has an incircle. Moreover, these incircles form a circle packing, since they touch in single points of appropriate edges. On the other hand, it is obvious that the polyhedron has no midsphere.

Combining theorems 1 and 3 the proof of Theorem 2 is very simple. Given a polyhedron P, let Q be the polyhedron isomorphic to P and midscribed to a sphere S with center O. Then the feet of the perpendiculars from O to (the planes of) the facets of Q are the centers of the incircles of the facets, hence belong to the relative interiors of the facets. Also, the feet of the perpendiculars to (the lines that carry) the edges of Q are the tangency points of the edges, therefore in the relative interior of the edges. It follows that all the conditions of Theorem 1 are satisfied, and hence Q and (a translate of) the polar Q^ of Q with respect to S are the bases of a 4–dimensional (p)-antiprism. This completes the proof of Theorem 2.

Since a (p)-antiprism it is an antiprism, Theorem 2 implies that every polyhedron is *isomorphic* to a basis of a 4-dimensional antiprism. However, this does not answer the following question:

Problem 2. Given a polyhedron P, is there a polyhedron Q dual to P such that P and Q are the bases of an antiprism?

A more specific question is

Problem 3. Given dual polyhedra P and Q. Under what conditions is there an antiprism with bases P and Q?

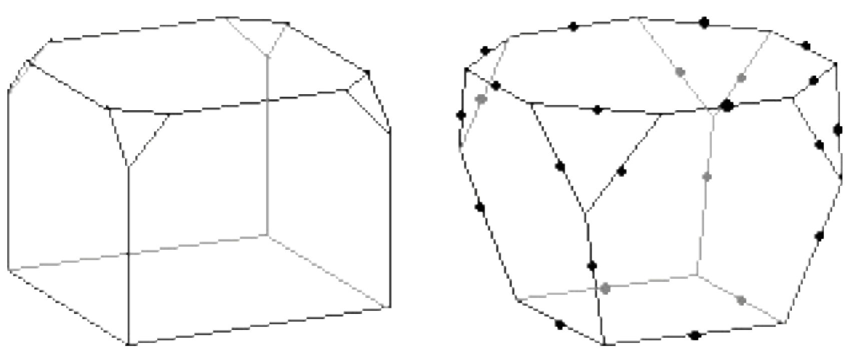

Figure 3. A polyhedron, and an isomorphic polyhedron that is midscribed; the tangency points are indicated by the solid dots.

4. Circle packings and midscribable polyhedra. A direct proof of Theorem 1 is presented in Broadie [2], but no analogously straightforward reference for Theorem 3 seems to exist. Instead, there are many papers dealing with a variety of topics, which need to be interpreted and combined appropriately in order to get the result. An account of the relevant works will be given here; for simplicity, it is convenient to introduce and additional concept.

Let G be a graph. We say that G has a *representation by a circle packing* on a sphere if there is a collection $\mathcal{C}$ of circles with dis-

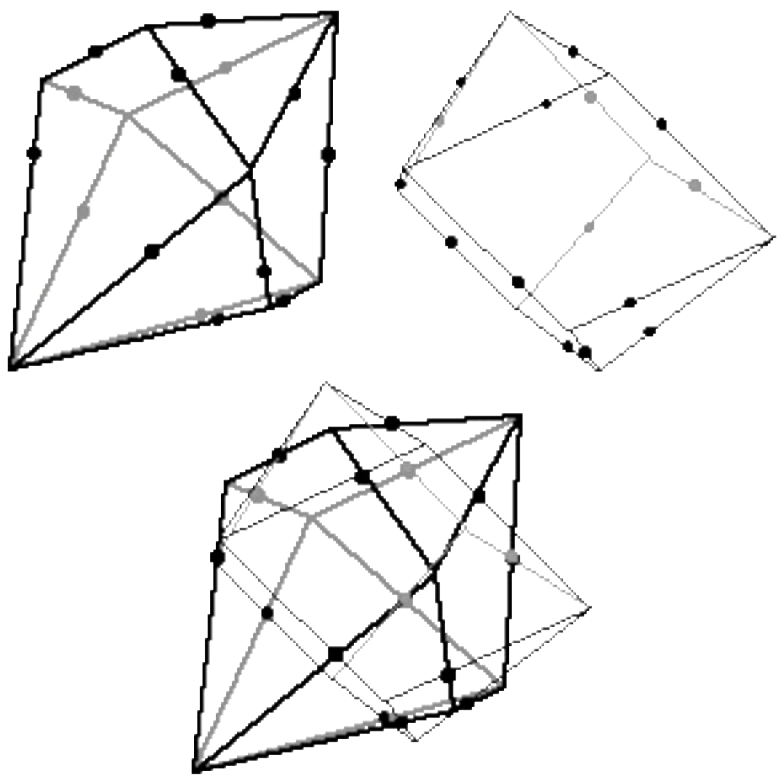

Figure 4. A midscribed polyhedron P (upper left) with the properties of Theorem 3, together with its polar (with respect to the midsphere) shown separately and also in the position in which its tangency points coincide with those of P. The edges meeting at each tangency point are perpendicular to each other.

joint interiors, each circle corresponding to a vertex of G, and two circles having a common point if and only if the corresponding vertices of G determine an edge of G.

It is immediate that if a polyhedron P has a midsphere S, then the incircles of the facets of P form a representation by a circle packing of the graph of vertices and edges of the polyhedron P^ polar to P with respect to S. But elementary properties of circles on spheres show that S is a midsphere of P^, hence the incircles of its faces form a representation by a circle packing of the graph of P. Using stereographic projections, these concepts and results can easily be modified to circle packings in the plane – from which the spherical versions can be retrieved as well.

These observations are important since the first relevant results were obtained by P. Koebe [7] in 1935. In an investigation of conformal maps of regions in the complex plane, he showed that the graph of every planar triangulation of a triangle has a representation by circle packings. At the end of his paper he mentions that he will soon present results that show the relevance of his theorems to the theory of polyhedra; however, no such paper appeared. According to Bieberbach [1], in a posthumous appreciation of Koebe's work, no traces of such a paper were ever found. In view of the facts presented in our note, it is reasonable to assume that Koebe may have had in mind the existence of a midsphere, or of incircles of facets, for polyhedra, or at least for simple polyhedra and for triangle-faced polyhedra.

W. Thurston [12] obtained in 1978/9, independently of Koebe a result on planar triangulations similar to Koebe's. This was contained in a set of lecture notes that were informally distributed, and that only in 2002 became generally available through the Internet [13]. Meanwhile, a number of people have devised ways of proving Theorem 3 in its general form. These papers base their proofs on various tools from the theory of functions of a complex variable, the geometry of the hyperbolic plane, critical points of nonlinear optimization problems, or other non-elementary considerations. None of these proofs is suitable for inclusion in the present paper. Among the available proofs most accessible are the ones given by Mohar & Thomassen [8, Section 2.8] and Ziegler [14, Lecture 1], and, for the triangle-faced polyhedra, Pach & Agarwal [9, Chapter 8]. Moreover, all presentations deal with the existence of circle packings or midspheres, but without the uniqueness part of Theorem 2. However, Schramm's paper contains no mention of centroids. The fact that the uniqueness part of theorem 2 can be derived from the results on circle packings was established recently by Springborn [11].

Another remarkable aspect of the topic is that the proofs of Theorem 2 are essentially existential, and do not provide a practical way of determining the midscribable polyhedron Q, that is isomorphic to a given polyhedron P. However, in an Internet posting, G. W. Hart [6] provides a very clever algorithm (in Mathematica®) that has as its input data about P, and provides data about Q and a drawing of it. Figure 4 has been drawn using a modification of Hart's software. The algorithm functions by successive approximations, and in general works very well. However, in some instances it fails to converge. The reason for this is not clear, and in all cases I checked one could reach the goal by taking as P an affine image of the original polyhedron.

5. Remarks and problems.

Theorem 3 implies a negative answer to Problem 6 of [5], by implying that every polyhedron P is isomorphic to a polyhedron Q such that corresponding edges of Q and a suitable dual of Q intersect in one and only one point. In fact, the edges may be required to intersect orthogonally.

One additional little-known consequence of Theorem 3 shuld be mentioned. Since the midscribed polyhedron Q isomorphic to a given polyhedron P is uniquely determined in the usual meaning of "uniquely" (that is, up to similarity), Q can be taken as the *ca-*

nonical representative of the isomorphism class of P (and of Q). While this is an interesting observation, I am not aware that it has been used to derive any significant results.

Concerning the topics of this paper, there are may open problems. Here are just a few.

It is easy to see that all convex 3-antiprisms with combinatorially equivalent bases are isotopic (that is, can be deformed into each other through polyhedra of the same combinatorial type.

Problem 4. Are all convex 4-antiprisms with combinatorially equivalent bases isotopic?

It follows from a theorem of Schulte [10] that there is no analogue of Theorem 3 in dimensions $d \geq 4$. But this still leaves open the question about the analogue of Theorem 2.

Problem 5. For $d \geq 4$, is every d-dimensional convex polytope P isomorphic to the basis of a (d+1)-antiprism?

Problem 6. If P is a non-convex polyhedron (in the sense of some more-or-less general definition), are there analogues of Theorems 2 and/or 3?

Finally, can one find e*lementary* proofs of Theorems 2 and 3?

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