Small configurations with many incidences

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A **configuration** (p_q, n_k) is a family of p points and n lines, such that each point is incident with (that is, belongs to) q of the lines, and each of the lines is incident with k of the points. If the number of points and lines is not relevant, we say that such a configuration is of **type** [q, k]. We shall mostly consider configurations in the Euclidean plane; in some cases we shall take them to be in the **extended Euclidean plane**. By this we mean the plane obtained by adding "ideal points" (also known as "points at infinity"), which are representatives of families consisting of all lines parallel to one direction. This plane is one of the ways of representing the real projective plane. It is well known that every configuration in the extended plane is equivalent (by a projective transformation) to a configuration in the Euclidean plane. The reason for the use of the extended Euclidean plane is that some configurations allow more symmetric representations in it than in the Euclidean plane itself.

Configurations of type [3, 3] have been studied for at least 130 years, and much has been written about them. Configurations of type [4, 4] have been studied for some twenty years; most of the known results are contained in [1], [3], [6], or can be traced through these papers. Some special cases of configurations of types [3, 4] and [4, 3] have also been investigated; see [4] for details and references.

It is easy to show that for every $q \geq 3$, $k \geq 3$ there exists configurations of type [q, k]. Indeed, starting with the lattice of integer points in the q-dimensional Euclidean space, consider a "box" of points each coordinate of which is an integer ≥ 0 and $< k$, together with lines through these points in the directions of the coordinate axes. Then we have a configuration $((k^q)_q, (q\ k^{q-1})_k)$ in the q-space; projecting it into a plane we obtain the desire configuration of type [q, k]. There are several other approaches to the construction of such configurations. The first to describe a (rather complicated) method was S. Kantor in 1879 (see [9]). However, all these constructions yield configurations with rather large numbers of points and lines, and their representation in the plane is quite jumbled. In fact, I have not seen any of these actually shown in a paper of book.

The only work which presents diagrams of some configurations of types [q, k] with $q \geq 4 \leq k$, $\max\{q,k\} \geq 5$ is the recent article [2]. The purpose of the present note is to show some "small" examples of such configurations. I hope that this may inspire others to find more and better examples, and ways of generating other families of such configurations. Some data are provided in Table 1.

Figure 1 shows the by now well-known configuration (21_4), first presented in [8]. (For configurations of type [k, k] it is customary to shorten the symbol (n_k, n_k) to (n_k).) It can be considered a *polycyclic* configuration in the terminology of [3], in that its points are situated at the vertices of regular polygons; in this case these are three heptagons. A notation for polycyclic configurations (n_4) is explained in [7].

k / q	4	5	6	7
4	**(21_4)**	**$(45_4,36_5)$**	**$(54_4,36_6)$**	**$(63_4,36_7)$**
5	**$(36_5,45_4)$**	**(50_5)**	**$(54_5,45_6)$**	$(126_5,90_7)$
6	**$(36_6,54_4)$**	**$(45_6,54_5)$**	(324_6)	
7	**$(36_7,63_4)$**	$(90_7,126_5)$		

Table 1. The smallest known configurations of type [q, k] for the indicated values of q and k. No reasonably small configurations of types not listed are known. The configurations in bold-face are conjectured to be smallest possible. Blank spaces mean I have no idea how to find a reasonably small configuration.

There are many different kinds of polycyclic configurations of type [4, 4]; some are important as starting points of "small" configurations of type [q, k] with larger values of the parameters. For example (see Figure 2) starting with a polycyclic configuration (36_4) with points at the vertices of four regular 9-gons, by adding the nine lines of mirror symmetry one obtains a configuration (36_5,45_4). Taking its polar yields a configuration (45_4, 36_5). A different configuration (45_4, 36_5) is obtained from the one in Figure 2 by adding the nine points at infinity.

Analogous constructions can be performed on the configuration in Figure 3. However, in this case there are additional collinearities among the points of the configuration. This allow us to add two families of nine lines each, and yields the configuration (36_6,54_4), shown in Figure 4. Adding to this the nine points at infinity yields a configuration (45_6,54_5). Adding instead the nine lines of mirror symmetry results in a configuration (36_7,63_4). Polars yield configurations (54_4, 36_6), (54_5, 45_6) and (63_4, 36_7), respectively. From two copies of (36_7,63_4) rotated (with respect to the common center) so

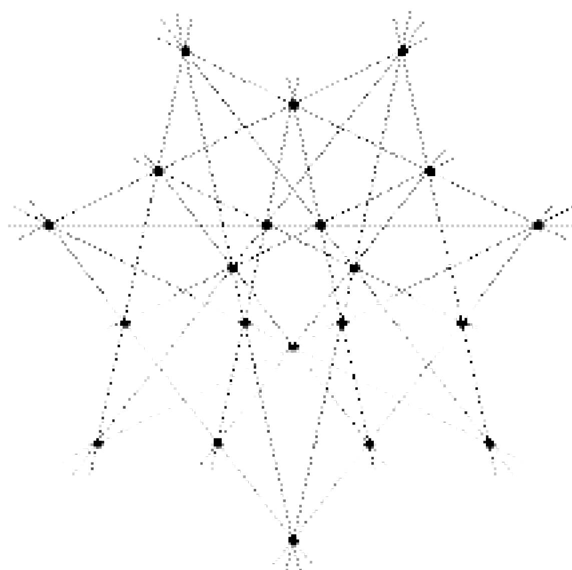

Figure 1. A configuration (21_4), with symbol 7#(3,2,1,3,2,1).

that the mirror lines of each copy are parallel to the non-mirror lines of the other, and then adding the 18 points at infinity, gives a configuration (90_7, 126_5). Taking six copies of (54_5, 45_6) suitably placed, and connecting them by 54 lines, yields a configuration (324_6). (Note that $324 = 6 \times 54 = 54 + 6 \times 45$.) The last two configurations are too large to present intelligibly in the available format, although there is no problem in drawing them in larger size.

In Figure 5 is shown a configuration (40_4). Adding the ten lines of mirror symmetry yields a configuration (40_5, 50_4). Adding instead the ten points at infinity gives a configuration (50_4, 40_5). Finally, adding both the lines of symmetry and the points at infinity results in a configuration (50_5).

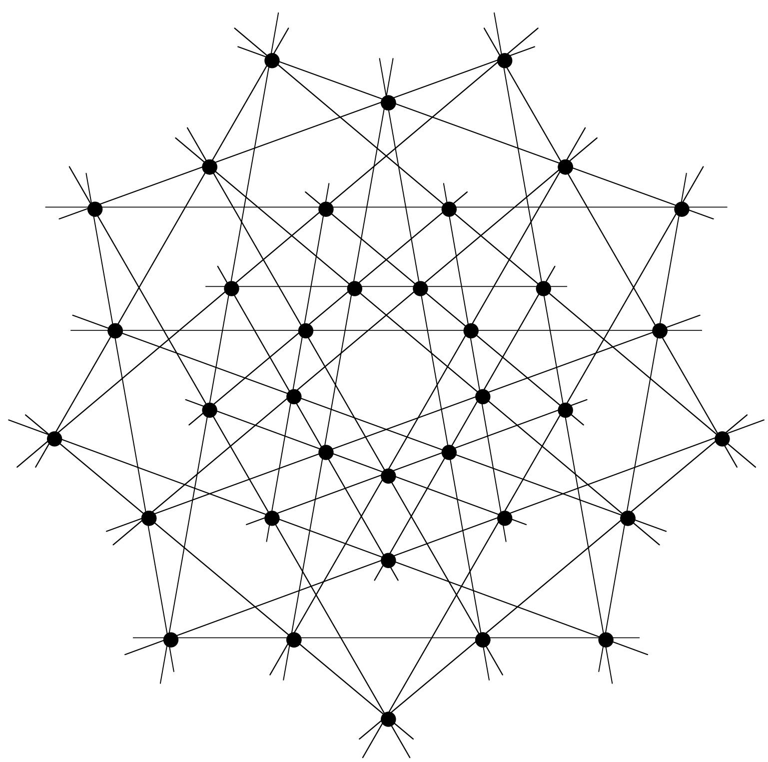

Figure 2. A configuration (36_4). In the notation of [7], this is the configuration 9#(4,3,1,3,1,3,2,1).

Two general remarks. First, all the configurations shown here are "genuine", that is, the incidences between point and lines that seem to be indicated by the diagrams are actual incidences. This can be confirmed by elementary calculations. Second, the constructions indicated can be performed on many other configurations (n_4), thus leading to infinitely many configurations of each of types [4, 5], [5, 4], [4, 6], [6, 4], [4, 7], [7, 4], [5, 5], [5, 6], [6, 5], and [6, 6]. Many of these are reasonably small; the ones presented are conjectured to be the smallest, as indicated in Table 1.

There are many unsolved problems related to the present topic. On the one hand, one may inquire whether there are

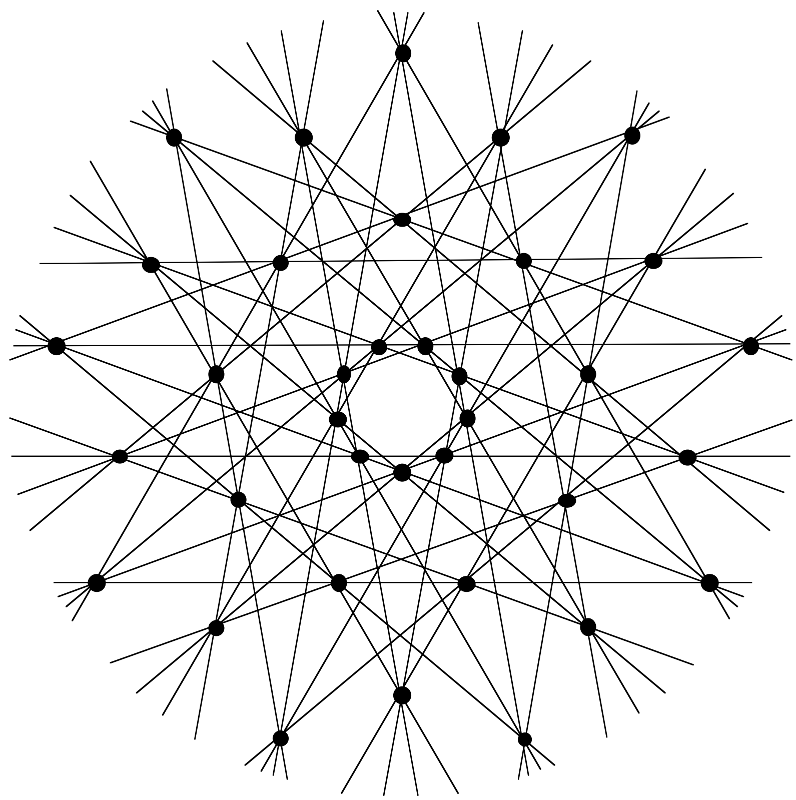

Figure 3. Another configuration (36_4). In the notation of [7], this is the configuration 9#(4,2,1,4,3,1,2,3).

possibilities of construction of reasonably small configurations of type [q, k] with q and k beyond the range of Table 1. On the other hand, if one allows configurations in which "lines" are replaced by "pseudolines" (that is, curves that behave globally, and pairwise among themselves, like lines), then smaller configurations of the types considered here become possible. As an example, in [5] a pseudoline configuration (180_6) is shown. As another example, Figure 6 shows a (22_4) configurations of pseudolines. It has been conjectured (see [6], [7]) that there is no configuration (22_4) of points and *lines*.

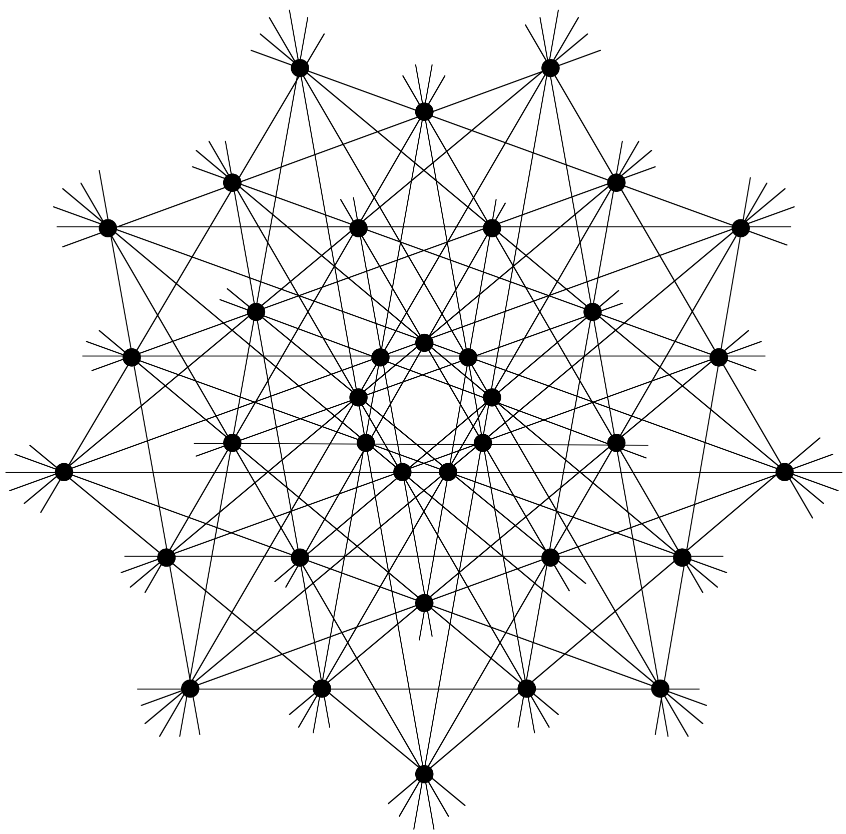

Figure 4. A configuration $(36_6,54_4)$, obtained by adding two families of nine lines each to the configuration in Figure 3.

Similar question may be posed for *combinatorial configurations*, in which "points" are just symbols, and "lines" are collections of symbols. It appears that for every q and k, there are combinatorial configurations that are smaller than the geometric ones considered here.

References

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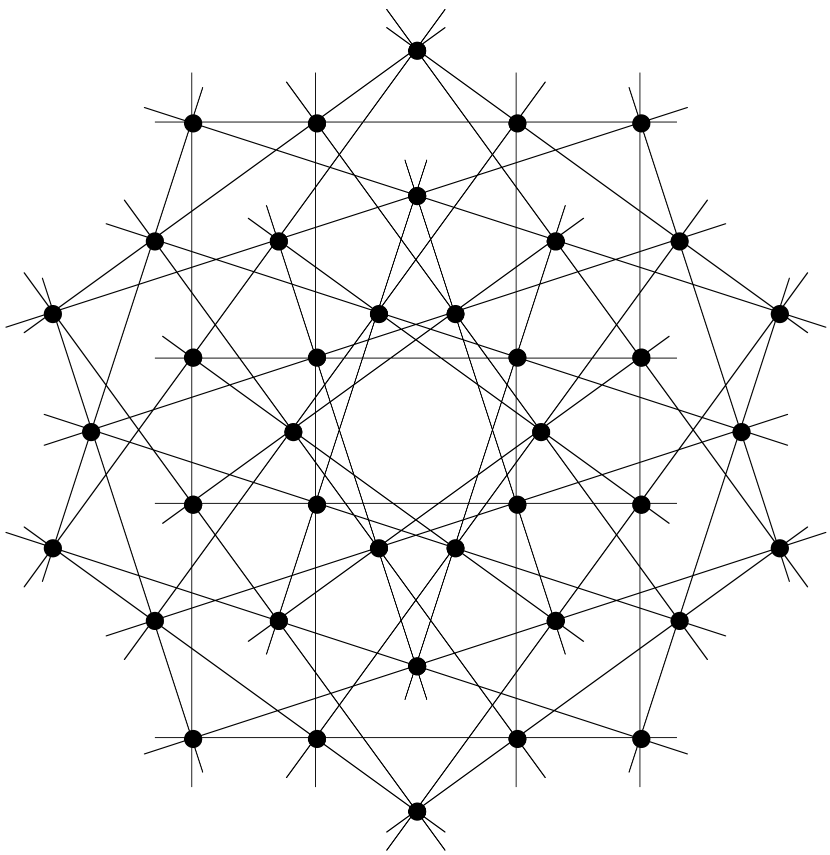

Figure 5. A configuration (40_4), with symbol 10#(4,3,2,4,1,2,3,1).

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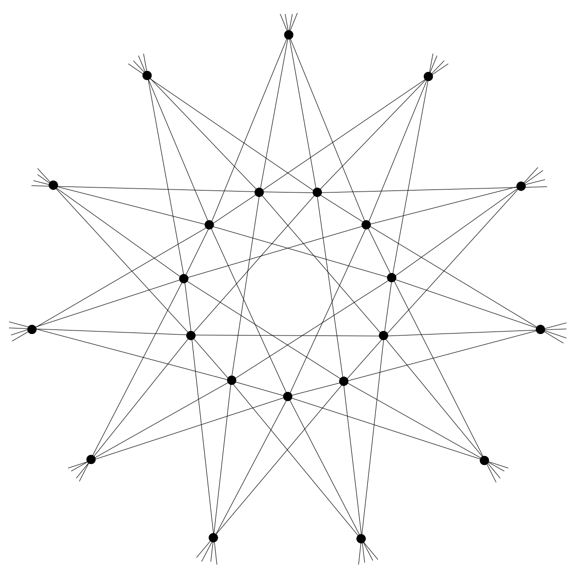

Figure 6. A (22_4) configuration of pseudolines. (It barely fails to be a configuration of lines!)