almost all n -- an update

by Branko Grünbaum
University of Washington, Box 354350, Seattle, WA 98195
e-mail: grunbaum@math.washington.edu

An **(n_k) configuration** is a family of n points and n (straight) lines in the Euclidean plane such that each point is on precisely k of the lines, and each line contains precisely k of the points. A configuration is said to be **connected** if it is possible to reach every point starting from an arbitrary point and stepping to other points only if they are on one of the lines of the configuration.

It has been known since the late 1800's that connected (n_3) configurations exist if an only if $n \geq 9$. However, the analogous question concerning connected (n_4) configurations was not considered till very recently ([2], [3]). This is probably due to the fact that — except for one paper early in the 20th century [5], there has been no mention in the literature of (n_4) configurations till 1990 ([4]).

In [3] it was shown that connected (n_4) configurations exist for all $n \geq 21$, with the possible exception of certain 32 values of n, the largest of which is $n = 179$. Since then, it was established that (n_4) configurations exist for more than half of these values, and the aim of the present note is to prove the following:

Theorem. Connected (n_4) configurations exist for all but finitely many values of n. Specifically, there are (n_4) configurations for all $n \geq 21$ except possibly if n has one of the following ten values: 22, 23, 26, 29, 31, 32, 34, 37, 38, 43.

For the proof we shall assume that the reader has access to [3], and we shall not repeat the arguments given there. In particular, we take as established that all configurations (n_4) with $n > 210$

as given the configurations (28_4) and (35_4), *ad hoc* constructions of which were provided in [4].

The first two new constructions (Figures 1 and 2) we need are based on ideas communicated to me by Prof. T. Pisanski, to whom I am obliged for permission to include them here. The first yields a configuration (25_4), the second a configuration (49_4).

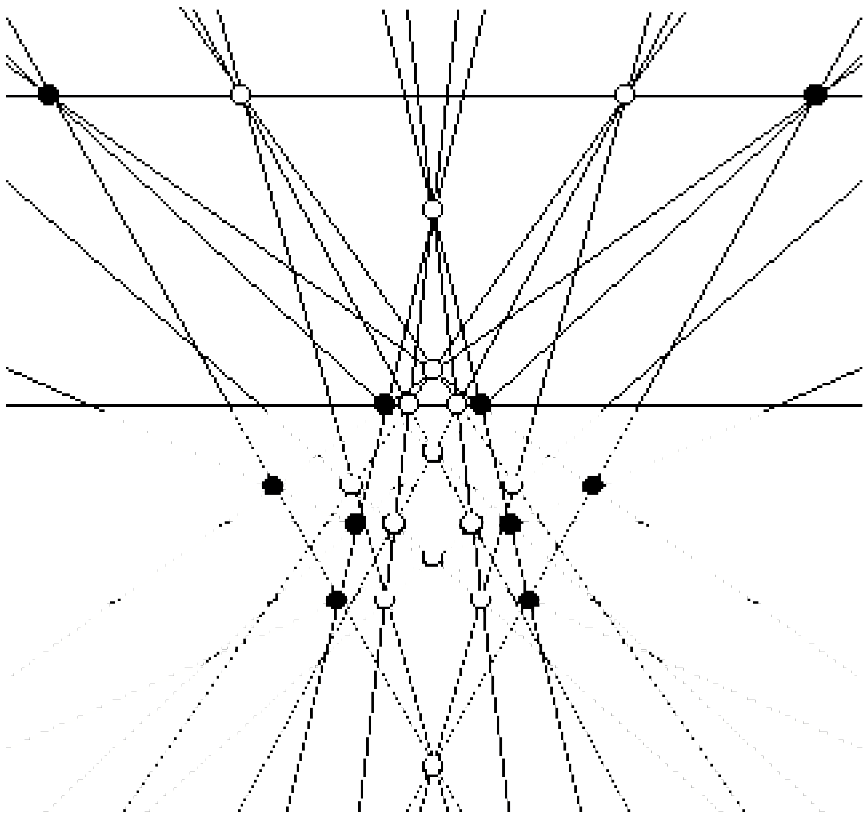


Figure 1. The solid dots indicate the points of a (10_3) configuration, symmetric with respect to a vertical mirror. Taking it and a copy shrunk towards that mirror, together with the five intersection points on the mirror (hollow dots), and the five lines perpendicular to the mirror, yields a (25_4) configuration.

the second construction from [3]. It is most easily visualized and explained in 3-space; obviously, once the new configuration is constructed, it is easy to project it into the plane. We start with an (m_4) configuration C in the plane. We assume that this is the (x,y)-plane in a Cartesian (x,y,z)-system of coordinates, and that C has $p \geq 1$ lines parallel to the x axis, and $q \geq 1$ lines parallel to the y axis. Note that by an affine transformation (which does not

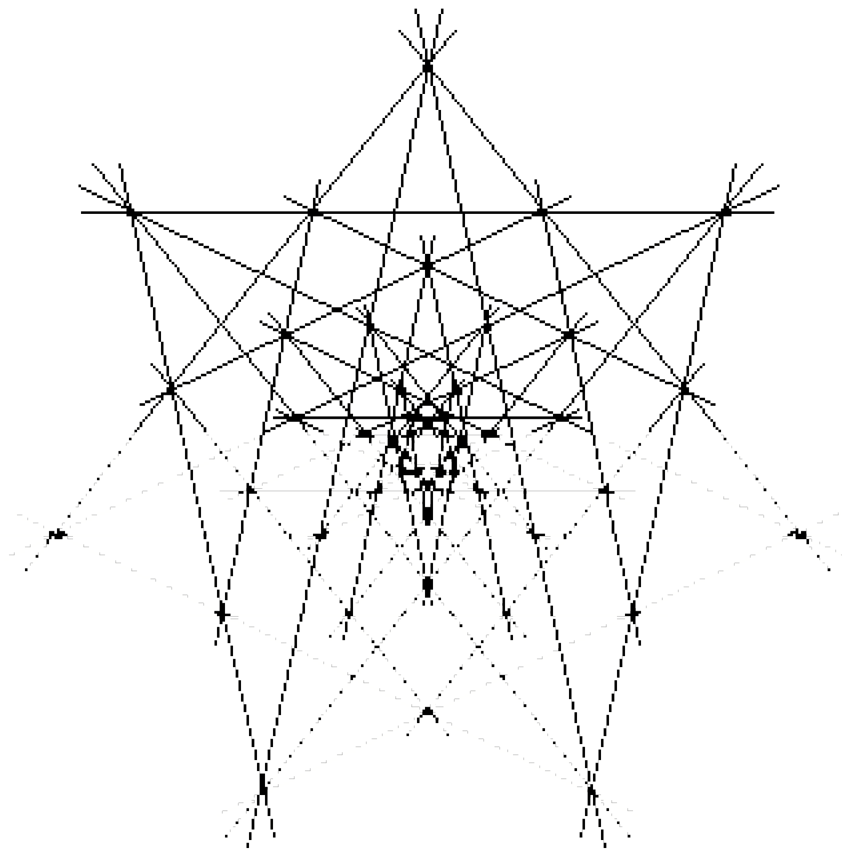


Figure 2. A (49_4) configurations. In the notation used in [3], this is the configuration $7\#2_1 2_1 3_2 3_2 1_2 1_3\ 3_2$.

orthogonal. We select a real number $h > 1$ and keep it constant throughout the discussion. We construct two copies of $\mathcal{C}$. One is $\mathcal{C}'$, obtained from $\mathcal{C}$ by stretching $\mathcal{C}$ in ratio $(h-1)/h$ (that is, in fact, shrinking it) towards the y-axis, stretching it in ratio $(h+1)h$ towards the x-axis, and then translating it to the level $z = 1$. The other is $\mathcal{C}''$, obtained similarly but by using the ratio $(h+1)/h$ for stretching towards the y-axis, $(h-1)h$ for the ratio towards the x-axis, and translation to the plane $z = -1$. Thus, $\mathcal{C}'$ is obtained from $\mathcal{C}$ by the map $f(x, y, 0) = (x(h-1)/h, y(h+1)/h, 1)$, and $\mathcal{C}''$ by $g(x, y, 0) = (x(h+1)/h, y(h-1)/h, -1)$. It is easy to check that for each point $A = (x, y, 0)$ the points A, $f(A)$ and $g(A)$ are collinear, and that the points $h(A) = (0, 2y, h)$ and $h^*(A) = (2x, 0, -h)$ are collinear with them. Now, for any four points A_j $(j = 1, 2, 3, 4)$ of $\mathcal{C}$ that are on a line L parallel to the x-axis — that is, have the same y-coordinate — the point $h(A_j)$ will be the same since it does not depend on the x-coordinate. Therefore we can conclude that by deleting the line L from the configuration $\mathcal{C}$ and its parallels in $\mathcal{C}'$ and $\mathcal{C}''$, while adding the lines from A_j to $h(A_j)$, the points A_j and the corresponding points in $\mathcal{C}'$ and $\mathcal{C}''$ will remain incident with four lines, and the new point $h(A_j)$ will also be incident with four lines. We deleted three lines and added four, and also added one point. Thus, from an (m_4) configuration we obtained a configuration (n_4) where $n = 3m+1$. Analogously, any four points of $\mathcal{C}$ collinear on a line parallel with the y-axis may lead to an additional increase in the number of points and lines. Proceeding similarly with some or all lines parallel to either the x-axis or the y-axis, we see that from (m_4) we can obtain confi-gurations (n_4) for each n such that $3m + 1 \le n \le 3m + p + q$.

can use the “polycyclic” configurations of the kind described in [2], either directly or by applying to them the “triplication” construction given above. The relevant details concerning the configurations (n_4) to which the “triplication” is applied are as follows.

For $n = 3m$ with $m \geq 7$ we use $m\#2_1 3_2\ 3_1$; it has two families of three parallel lines each for odd m, and $4 + 4$ lines for even m; hence $p + q$ is 6 or 8, respectively.

For $n = 4m$ with $m \geq 9$ use $m\#2_1 3_2 4_3\ 4_1$; then $p + q$ is 8 for odd m and 16 for even m.

The last construction we need (“duplication”) is best explained on hand of Figure 3. We start with a “polycyclic” configuration (27_4), namely $9\#3_2 1_3\ 1_2$ shown in the top part of Figure 3. In the bottom part of the figure is a copy of the same con-figuration, but shrunk in a suitable ratio towards the vertical symmetry line. Both configurations have also been slightly modified. In the top copy, one line is shown by a dotted line; this is supposed to indicate that this line is omitted in the construction of a configuration (53_4). In the bottom copy one point has been singled out, and longer segments of the lines throught it have been drawn; this point is also omitted in the construction. The specific ratio of shrinking, and the positioning of the two copies, are such that the four lines just mentioned pass through the four points incident with the line omitted from the top part. In this way, each of the 53 lines of the constructed object is incident with four points, and each of the 53 points is incident with four lines — hence we have constructed a (53_4) configuration.

guration which has a point such that the four lines passing through it are incident with the four points of some other line. Since there is

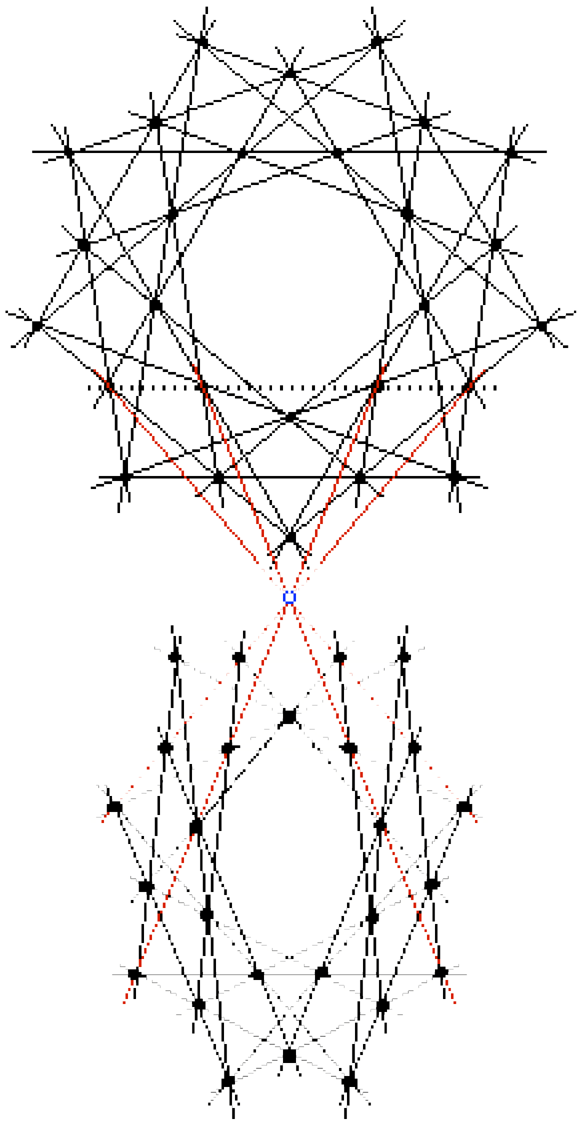
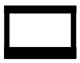


Figure 3. The “duplication” construction applied to a configuration (27_4) yields a configuration (53_4).

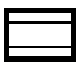
transformation, it is easy to avoid unwanted incidences between the points and lines of the two copies. It can also be applied to a pair of different configurations, provided they have appropriate lines and points.

The configurations obtained by the various methods are specified in Figure 4. An explanation of the backgrounds follows:

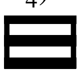
 indicates configurations obtained by the first construction explained in [3]. Except for the first few, they could also be obtained as "polycyclic" configurations with five orbits.

 indicates "polycyclic" configuration with three orbits.

 indicates "polycyclic" configuration with four orbits.

 indicates that the configuration is obtained by the "triplication" construction; the numeral indicates the starting configuration of the construction.

 indicates that the configuration is obtained by the "duplication" construction. A (61_4) configuration is obtained by applying the duplication construction twice to a (21_4) configuration, and a (62_4) by using the construction on a (21_4) and a (42_4).

 indicates a variant of the "duplication" construction with the two copies concentric, and two lines and two points deleted.

 indicates a special construction, explained here or in [4].

 indicates that no configuration (n_4) is known.

Remarks.

1. It is not clear whether one should expect that the configu-rations "missing" in the theorem actually exist, or not, but it is clear that some new ideas will be required to settle each of these cases.

2. Combinatorial configurations (n_4) exist for all $n \geq 13$, and those with $n = 13$ or 14 are not geometrically realizable (see [5]). The realizability is undecided for $15 \leq n \leq 20$; it may be conjectured that no geometric configurations of this kind exist.

3. The "polycyclic" configurations have been studied in considerable detail in [1].

4. As mentioned in [3], the known constructions of connected (n_k) configurations are sufficient to show that for each k, there exists an $N(k)$ such that all $n \geq N(k)$ admit such configurations. However, the values of $N(k)$ are very large, and the present methods of construction of such configurations seem not adequate to substantially fill the gaps even for $n = 5$.

References.

[1] M. Boben and T. Pisanski, Polycyclic configurations. Preprint, August 2001.

[2] B. Grünbaum, Which (n_4) configurations exist ? Geombinatorics 9(2000), 164 - 169.

[3] B. Grünbaum, Connected (n_4) configurations exist for almost all n. Geombinatorics 10(2000), 24 - 29.

[4] B. Grünbaum and J. F. Rigby, The real configuration (21_4). J. London Math. Soc. (2) 41 (1990), 336 - 346.

[5] E. Merlin, Sur les configurations planes n_4. Bull. Cl. Sci. Acad. Roy. Belg. 1913, pp. 647 – 660.

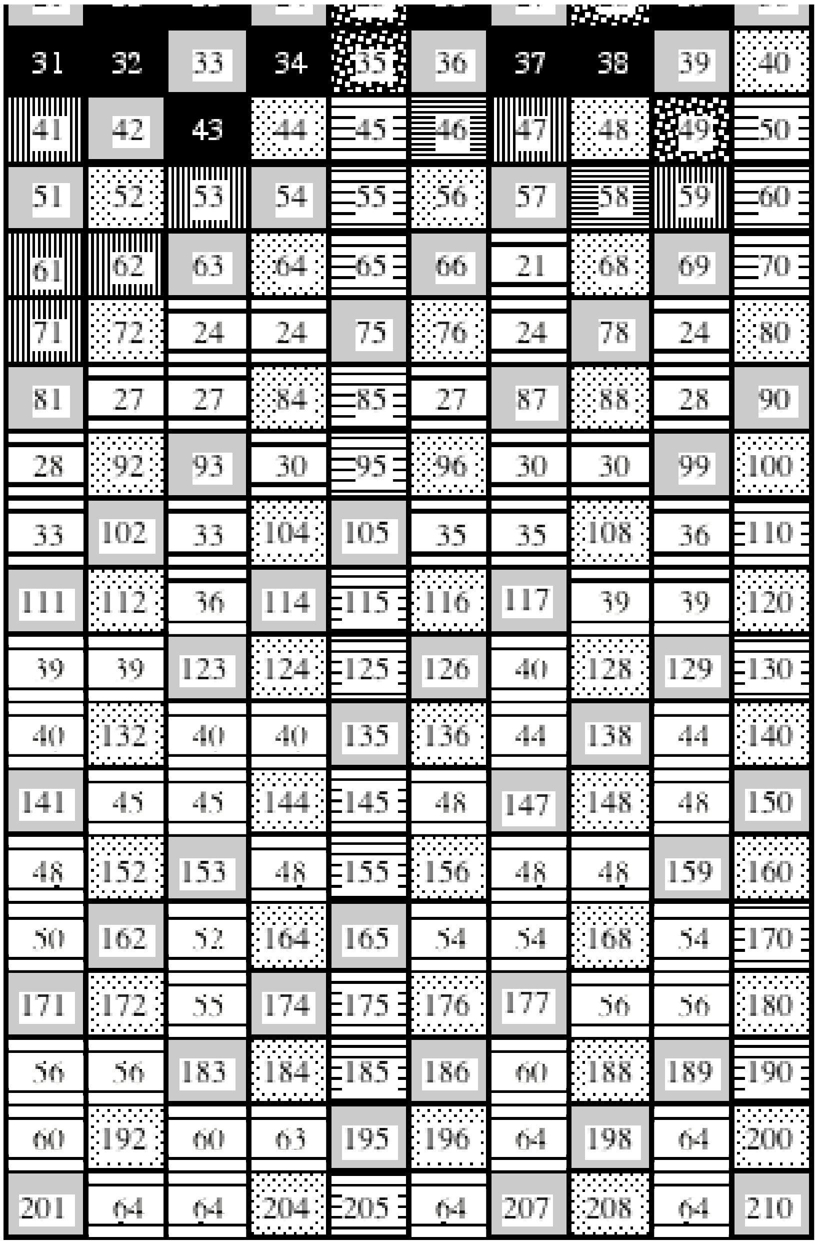


Figure 4. A guide to the construction of the configurations (n_4).