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Connected (n_4) configurations exist for almost all n

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Dedicated to Aryeh Dvoretzky, who taught me to respect formalities but look for content

This short note is a continuation of [2]. Its main goal is to strenghten the result of that note, by showing how to construct connected (n_4) configurations for all values of n with a small number of exceptions (only five of which exceed 100). In order to make this note more selfcontained, we repeat a number of explanations from the earlier one.

An **(n_4) configuration** is a family of n points and n (straight) lines in the Euclidean plane such that each point is on precisely four of the lines, and each line contains precisely four of the points. A configuration is said to be **connected** if it is possible to reach every point starting from an arbitrary point and stepping to other points only if they are on one of the lines of the configuration. In the earlier note it was established that there exist connected (n_4) configurations for all $n \geq 21$ except possibly in the following cases: $n = 32$ or $n = p$ or $n = 2p$ or $n = p^2$ or $n = 2p^2$ or $n = p_1p_2$, where p, p_1, p_2 are odd primes and $p_1 < p_2 < 2p_1$.

The configurations constructed in [2] are particularly nice since they can be drawn with a high degree of symmetry. For the present strengthening of this result we shall need configurations with much less symmetry.

Theorem. Connected (n_4) configurations exist for all but finitely many values of n. Specifically, there are (n_4) configurations for all $n \geq 21$ except possibly if n has one of the following thirty two values: 22, 23, 25, 26, 29, 31, 32, 34, 37, 38, 41, 43, 46, 47, 49, 53, 58, 59, 61, 62, 67, 71, 77, 79, 89, 97, 98, 103, 113, 131, 178, 179.

We note that the above list starts with $n = 21$ since this is the smallest value for which an (n_4) configurations is known to exist. It is conjectured that no such configuration exists for smaller n. However, this conjecture has not been established for $15 \leq n \leq 20$.

For the proof we need two constructions of (n_4) configurations; one was briefly mentioned in Remark (i) of [2], the other is apparently new.

The first construction starts with an arbitrary (k_3) configuration C, that is, a family of k points and k lines in the plane, with three of the points on each of the lines, and three lines through each of the points. We select in the plane a line L which misses all the points of C and is neither parallel nor perpendicular to any line determined by any two points of C. We construct three additional copies of C by stretching C through three different ratios in the direction perpendicular to L. The resulting configuration C^* consists of the four replicas of C, together with the k intersection points of C with L (which are also intersection points with L of the copies of C), and of the k lines perpendicular to L which pass through the points of C (and of the other copies). Hence this construction yields a configuration (n_4) with $n = 5k$. Since (k_3) configurations are well-known to exist if and only if $k \geq 9$, this establishes the existence of configurations (n_4) for all $n \geq 45$ which are divisible by 5. Very important for the sequel is the observation that, as follows from the construction, each such configuration contains a set of k parallel lines. The construction is illustrated in Figure 1, where C is the heavily drawn (9_3) configuration, and L is the dashed horizontal line. In order to reduce clutter, only one of the three stretched versions of C is shown (thin solid lines). The new points are the solid black ones on L, and the new lines are the vertical dashed ones. They contain only two points, since two of the copies of C are nor shown; for the same resason, the solid dots on L are on only two visible lines of the configuration.

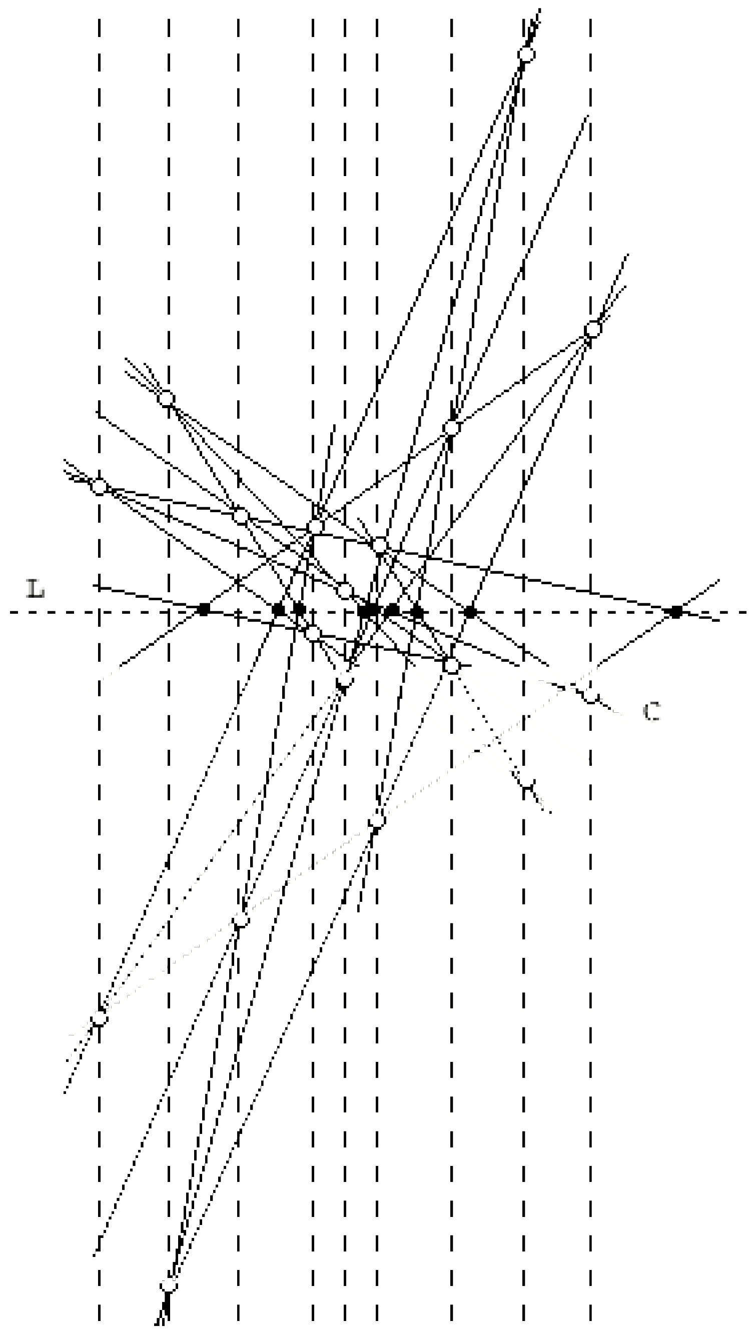

Figure 1.

The second construction has the interesting feature that it is more easily visualized and explained in 3-space; the resulting configuration is then readily projected into the plane. We start with an (m_4) configuration C in the plane. Let us assume that this is the (x,y)-plane in a Cartesian (x,y,z)-system of coordinates, and that C has p lines parallel to the x axis. We select a real number $h > 1$ and keep it constant throughout the discussion; it is convenient (but not necessary) to think of $h = 10$. We construct two copies of C. One is C', obtained from C by stretching C in ratio $(h-1)/h$ (that is, in fact, shrinking it) towards the y-axis, and then translating it to the level $z = 1$. The other is C'', obtained similarly but by using the ratio $(h+1)/h$, and translation to the plane $z = -1$. Then it is immediately verified that if a point of C has coordinates $(a,b,0)$, the corresponding points of C' and C'' have coordinates $(a(h-1)/h, b, 1)$ and $(a(h+1)/h, b, -1)$. Moreover, these three points are on one line with the fourth point $H_b = (0, b, h)$. This is illustrated in Figure 2, which shows the (x,z)-plane only; we have assumed that one of the p parallel lines is contained in the (x,z)-plane, hence coincides with the x-axis. We obtain an (n_4) configuration with $n = 3m + q$ (where $1 \le q \le p$) if from C we delete q lines (from among the p parallel ones, and distinguished by differing values of b), delete from C' and C'' the corresponding lines, and introduce instead the $4q$ lines determined by the points on the deleted lines and the q points H_b, and also the points H_b themselves. (In Figure 2, the deleted lined are the horizontal ones, while the new lines are the slanted solid lines. The new point is H_0.) The resulting (n_4) configuration has $3m - 3q + 4q = 3m + q$ lines, and $3m + q$ points, as claimed.

Now it is a simple matter to show that (n_4) configurations exist for all $n \ge 210$. Indeed, by the first construction there is for each $k \ge 9$ a $((5k)_4)$ configuration with $p = k$ parallel lines. It follows by the second construction that for all $k \ge 9$ and $1 \le q \le k$ there exists a $((15k+q)_4)$ configuration. Since $15k = 5\cdot(3k)$, by the first construction we can add $q = 0$ to the range of q. Thus (n_4) configurations exist all values of n such that $15k \le n \le 16k$; for $k \ge 14$ these ranges are contiguous or overlapping, and so the claim is established.

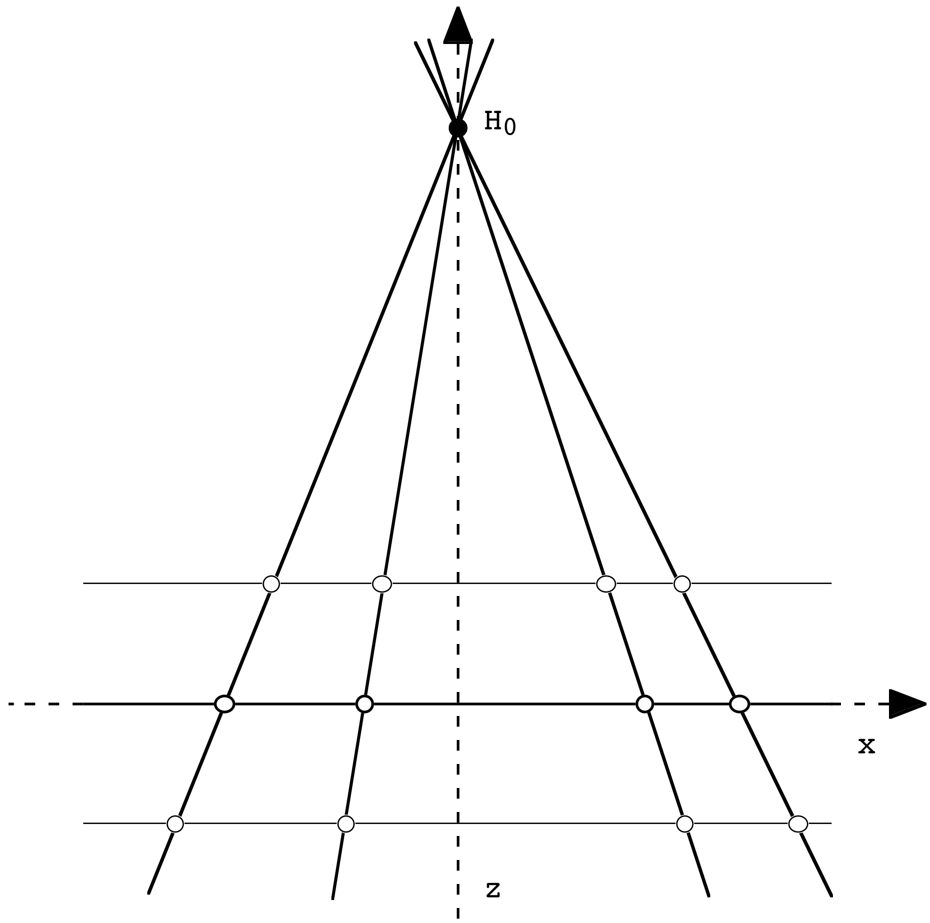

Figure 2.

To show the validity of the second part of the theorem, which gives the specific values of n for which it is at present not known whether an (n_4) configuration exist, we can use the various constructions described in [2] and its references, together with the second technique described above. Since there is no particular interest in these details we shall not give them here, but will gladly supply them to interested readers. We should only mention two facts useful in this context:

(i) The (n_4) configurations $m\#2_1 3_2 4_3 \ldots q_{q-1}\ q_1$, described in [2], have sets of at least q parallel lines each;

(ii) Using suitable projective transformations, every (n_4) configuration can be drawn in such a way that it has at least two parallel lines. ◊

Remarks.

(1) The first part of the Theorem easily generalizes to the statement: For every $r \geq 4$ there exists an $N = N(r)$ such that configurations (n_r) exist for each $n \geq N$. For the proof by induction on r one only needs to adapt the two constructions described above. The first one yields a configuration $(((r+1)s)_r)$ from any $(s_{(r-1)})$, while the second yields, for any (s_r) and for each q not exceeding the number of parallel lines in (s_r), a configuration $(((r-1)s+q)_r)$. For example, in case $r = 5$ the result of the Theorem implies that $N(5) \leq 1512$.

(2) The explanation given in Remark (vi) of [2], that the conjectured characterization of astral (n_4) configurations formulated in [1] implicitly assumes that the configurations considered are connected, is appropriate. However, the formulation of this correction may give the impression that the **only** astral configurations that are not connected arise by taking two concentric copies of a connected astral configuration and rotating one with respect to the other through an arbitrary angle. In fact, starting from any k concentric copies of any astral configuration (n_4) as described in Theorems 1 and 2 of [2], if they are rotated through such angles that the vertices coincide with those of two regular (nk/2)-gons, a disconnected astral (n_4) is obtained. Two concentric copies of such a configuration may still be rotated arbitrarily, yielding a wider class of astral configurations. The totality of configurations described by all these steps is the conjectured characterization of astral configurations (n_4).

References.

[1] B. Grünbaum, Astral (n_4) configurations. Geombinatorics 9(2000), 123 - 134.

[2] B. Grünbaum, Which (n_4) configurations exist ? Geombinatorics 9(2000), 164 - 169.